Supplementary Exercises on Limits

1. Show the following results:

(a) $\lim\limits_{n\to\infty} \dfrac{n^n}{(2n)!} = 0$ *(b) $\lim\limits_{n\to\infty} \dfrac{(2n)!}{a^{n!}} = 0$ where $a > 1$ (c) $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0$

(d) $\lim\limits_{n\to\infty} \dfrac{n^n}{(n!)^2} = 0$ *(e) $\lim\limits_{n\to\infty} \dfrac{(n!)^n}{n^{n^2}} = 0$

2. Compute the following limits using the Taylor series expansions:

(a) $\lim\limits_{x\to 0} \dfrac{2[\tan(x) - \sin(x)] - x^3}{x^5}$ (b) $\lim\limits_{x\to 0} \dfrac{\ln(1+x+x^2) + \ln(1-x+x^2)}{x(e^x - 1)}$

(c) $\lim\limits_{x\to\infty} \left\{x - x^2 \ln\left(1 + \dfrac{1}{x}\right)\right\}$ (d) $\lim\limits_{x\to 0} \left\{\dfrac{1}{x^2} - \cot^2(x)\right\}$

(e) $\lim\limits_{x\to 0} \left\{\dfrac{1}{x^2} - \dfrac{\cot(x)}{x}\right\}$ (f) $\lim\limits_{x\to 0} \left\{\dfrac{2 + \cos(x)}{x^3 \sin(x)} - \dfrac{3}{x^4}\right\}$

*(g) $\lim\limits_{x\to 0} \dfrac{x + \ln\left(\sqrt{1+x^2} - x\right)}{x^3}$

3. Expand the function given by $f(x) = \dfrac{1+x}{(1-x)^3}$ as a Taylor series in the neighbourhood of the point 0. Using this expansion, find the sum of the series

$$\sum_{m=1}^{\infty} \frac{m^2}{2^{m-1}}$$

Brief Solutions

2. (a) $\dfrac{1}{4}$ (b) 1 (c) $\dfrac{1}{2}$ (d) $\dfrac{2}{3}$ (e) $\dfrac{1}{3}$ (f) $\dfrac{1}{60}$

(g) $\dfrac{1}{6}$

3. 12