MT1: Tutorial solutions

23. Improper integration

1. (a) Indefinite integral is $\frac{1}{3}x^3 + C$: this tends to infinity as $x \to \infty$, so the integral does not converge.

   (b) Here the $w^{-1/4}$ term has a singularity at $w = 0$.

$$F(w) = \int \left(w^2 + w^{-1/4}\right) \mathrm{d}w = \frac{1}{3}w^3 + \frac{4}{3}w^{3/4} + C$$

   Hence

$$I = \lim_{W \to 0} [F(1) - F(W)] = F(1) = 5/3$$

   (c) It may be easiest here to make the substitution $x = 1 - y$: then

$$I = \int_1^0 \frac{1}{x}(-1)\mathrm{d}x = \int_0^1 \frac{1}{x}\mathrm{d}x$$

   This has a singularity at $x = 0$ and, since the indefinite integral is $\ln(x)$ which is not defined at $x = 0$, the integral does not converge.

   (d)

$$F(x) = \int e^{-x}\mathrm{d}x = -e^{-x} + C$$

   So

$$I = \lim_{X \to \infty} F(X) - F(0) = 0 + e^0 = 1$$

   (e)

$$F(x) = \int \sin(x)\mathrm{d}x = \cos x + C$$

   Now

$$\lim_{x \to \infty} \cos(x)$$

   is not defined... cos does not tend to a value but carries on cycling round between 1 and -1. So this integral is not defined.

   (f) This is a doubly improper integral; the integrand has a singularity at $q = 0$ and the upper limit is infinity. The easiest way to deal with this is to split it into two integrals at some point where the integrand clearly is defined, e.g.

$$\int_0^\infty \frac{1}{q^2}\mathrm{d}q = \int_0^1 \frac{1}{q^2}\mathrm{d}q + \int_1^\infty \frac{1}{q^2}\mathrm{d}q$$

The original integral is defined iff both the two new integrals are defined. Now

$$\int_0^1 \frac{1}{q^2}\mathrm{d}q$$

is one that we looked at in the lectures:

$$F(q) = \int \frac{1}{q^2}\mathrm{d}q = \frac{1}{q} + C$$

and

$$\lim_{q \to 0} F(q) = \lim_{q \to 0} \frac{1}{q} = \infty$$

So this integral is not defined, and we needn't worry about the other one; the overall integral does not have a defined value.

(g) The integrand has a singularity at $x = 3$.

$$F(x) = \int \frac{1}{\sqrt{3-x}}\mathrm{d}x = -2\sqrt{3-x} + C$$

so

$$I = \lim_{X \to 3} F(X) - F(0) = 0 + 2\sqrt{3} = 2\sqrt{3}$$

2.

$$W = \int_R^\infty \frac{Gm_1m_2}{r^2}\mathrm{d}r$$

The indefinite integral is

$$F(r) = -\frac{Gm_1m_2}{r}$$

So

$$W = \lim_{r \to \infty} F(r) - F(R)$$

But the limit is clearly zero, so

$$W = -F(R) = \frac{Gm_1m_2}{R}$$

3. (a) Follow the hint:

$$I = \int \frac{1}{(x^2+2)^{3/2}}\mathrm{d}x = \int \frac{1}{(2\tan^2\theta + 2)^{3/2}}\mathrm{d}x = \int \frac{1}{[2(1+\tan^2\theta)]^{3/2}}\mathrm{d}x$$
$$= \int \frac{1}{(2\sec^2\theta)^{3/2}}\mathrm{d}x = \int \frac{1}{2^{3/2}\sec^3\theta}\mathrm{d}x$$

Now we need to use

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{2}\sec^2\theta$$

so that

$$\mathrm{d}x = \sqrt{2}\sec^2\theta\mathrm{d}\theta$$

and then

$$F(x) = \int \frac{1}{2^{3/2}\sec^3\theta}\sqrt{2}\sec^2\theta\mathrm{d}\theta = \frac{1}{2\sec\theta}\mathrm{d}\theta = \frac{1}{2}\cos\theta\mathrm{d}\theta = \frac{1}{2}\sin\theta + C$$

Now we're told that $\sin\theta = x/(x^2+2)^{1/2}$ (proof: draw a right-angled triangle with adjacent side length $\sqrt{2}$ and opposite side x, giving $\tan\theta = x/\sqrt{2}$: then you have $\sin\theta$). So

$$F(x) = \frac{1}{2}\sin\theta + C = \frac{x}{2(x^2+2)^{1/2}} + C$$

QED.

(b) Split this up into two integrals:

$$I = \int_{-\infty}^{+\infty} \frac{1}{(x^2+2)^{3/2}}\mathrm{d}x = \int_0^{\infty} \frac{1}{(x^2+2)^{3/2}}\mathrm{d}x + \int_{-\infty}^{0} \frac{1}{(x^2+2)^{3/2}}$$

Now the first integral gives

$$I_1 = \lim_{x\to\infty} F(x) - F(0)$$

$F(0)$ is 0, but $F(x)$ tends to 1/2 as $x\to\infty$. So $I_1 = 1/2$. By symmetry (or find the limit if unsure!) $I_2 = 1/2$ as well. So the integral is 1.

4. Follow the hint and try this by parts; for definite integrals, the integration by parts rule is

$$\int_a^b uv'\mathrm{d}x = [uv]_a^b - \int_a^b u'v\mathrm{d}x$$

Let $u = x^n$ and $v' = e^{-x}$; this means that $u' = nx^{n-1}$ and $v = -e^{-x}$. Then

$$\int_0^{\infty} x^n e^{-x}\mathrm{d}x = \left[-e^{-x}x^n\right]_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}\mathrm{d}x$$

Now the definite integral (first term on the RHS, in square brackets) needs to be done by taking the limit; it is zero at $x = 0$ because of the x^n term, and the limit as $x\to\infty$ is also zero because of the e^{-x} term. So we have

$$\int_0^{\infty} x^n e^{-x}\mathrm{d}x = n\int_0^{\infty} x^{n-1}e^{-x}\mathrm{d}x$$

But now we can repeat the process, by integrating the RHS by parts again!

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n \int_0^\infty x^{n-1} e^{-x} \mathrm{d}x = n(n-1) \int_0^\infty x^{n-2} e^{-x} \mathrm{d}x$$

etc. Note that the index in the integral on the RHS goes down by one every time we integrate by parts again. When it's zero, we have

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n(n-1)(n-2) \ldots \int_0^\infty e^{-x} \mathrm{d}x$$

and the integral is one that we have done already above, with value 1. So we have

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = (n)(n-1)(n-2) \ldots (2)(1) = n!$$

QED.

Note that this problem has the interesting feature that you can't write down the indefinite integral...