SECTION C **Further Fourier Series**

By the end of this section you will be able to

- obtain the Fourier series for more complicated functions
- visualize graphs of Fourier series

In the last section we found Fourier series of rectangular waveforms. In this section we will obtain Fourier series of other waveforms such as sawtooth. The reason for placing non-rectangular waveforms in this section is because it is more difficult to find the Fourier series for such functions. The integration is more complicated, it generally involves the technique of 'integration by parts'. You need to be very familiar with this method otherwise you will find the C1 section tough going. However we do return to rectangular waveforms in section C3.

C1 **Fourier Series of Sawtooth Waveforms**

The integration by parts formula is given by

$$\int \left(uv'\right)\mathrm{d}x = uv - \int \left(u'v\right)\mathrm{d}x$$

The verbal form of integration by parts is stated as 'differentiate one (u) and integrate the other (v').'

As stated in the last section, obtaining a Fourier series means that you will need to have a thorough understanding of algebra, trigonometry, integration and sketching graphs. Application of integration by parts formula should be at your finger tips.

For Example 5 we will use the following well known trigonometric result:

(*) $\sin\left(2\pi k\right) = 0$ where k is a whole number.

This means $\sin\left(\text{even multiple of } \pi\right) = 0$ which we can visualize from the graph:

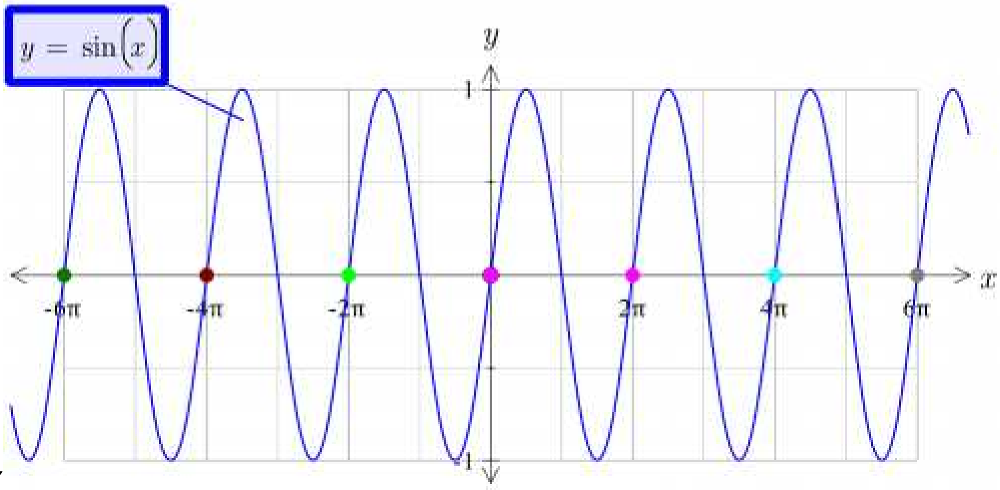

Figure 17

Also

(**) $\cos(2\pi k) = 1$ where k represents a whole number.

This means $\cos(\text{even multiple of } \pi) = 0$ which can be visualized from the graph:

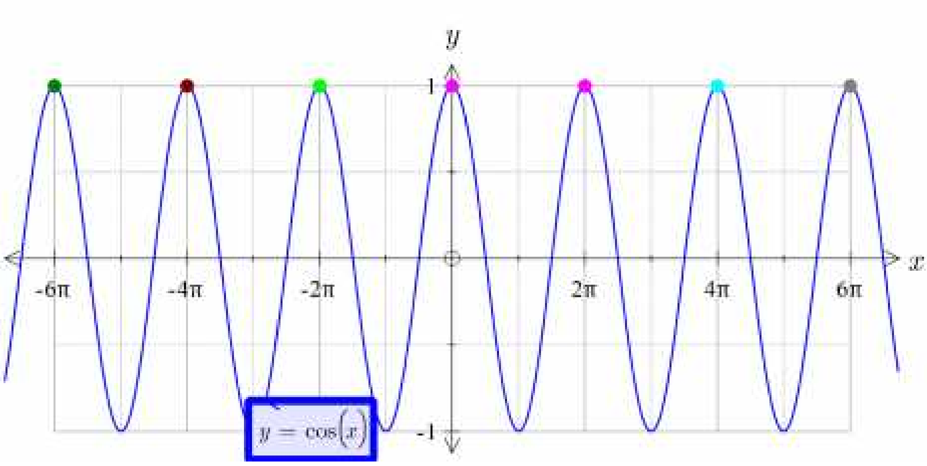

Figure 18

Example 5

Determine the Fourier series of the sawtooth waveform $f(t)$ shown in Fig. 19:

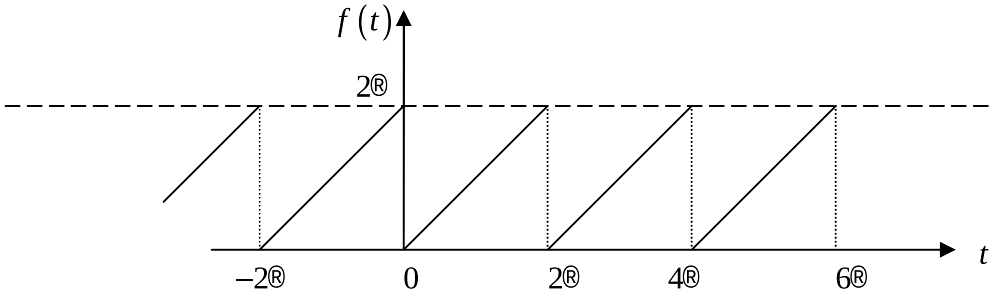

Figure 19

Solution

How do we find the Fourier series of $f(t)$?

Since $f(t)$ has period 2π so we obtain the values of the Fourier coefficients A_0, A_k and B_k given by the formulae from the last section:

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$ [Average value of $f(t)$]

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$ [These are the Cosine coefficients]

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$ [These are the Sine coefficients]

What are we going to use for $f(t)$ in these formulae?

We examine the graph between 0 and 2π. *Why?*

Because the period of the given function is 2π and we only need to cover one complete period of the function.

Since the graph shown in Fig. 19 is a straight line between 0 and 2π, we know it is of the form

$$y = mt + c$$

where m is the gradient and is given by

$$m = \text{gradient} = \frac{\cancel{2\pi}}{\cancel{2\pi}} = 1 \qquad \left[\text{Cancelling}\right]$$

What does c represent in $y = mt + c$?

It is the value where the line crosses the vertical axis. Hence from the graph we have $c = 0$.

Substituting these values, $m = 1$ and $c = 0$, into $y = mt + c$ gives

$$f\left(t\right) = y = t$$

Therefore the equation of the line between 0 to 2π is given by $f\left(t\right) = t$.

Also by observing the graph we can say that $f\left(t\right)$ has a period of 2π because it repeats itself every 2π interval. Let's find the average value A_0 first. The formula is

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(t\right)\,\mathrm{d}t$$

By examining the graph below, the integral $\int_{-\pi}^{\pi} f\left(t\right)\,\mathrm{d}t$ represents the area between the line $f\left(t\right)$ and the t axis from $-\pi$ to π. This area is the same as the area under the line $f\left(t\right) = t$ from 0 to 2π as shown shaded in Fig. 20.

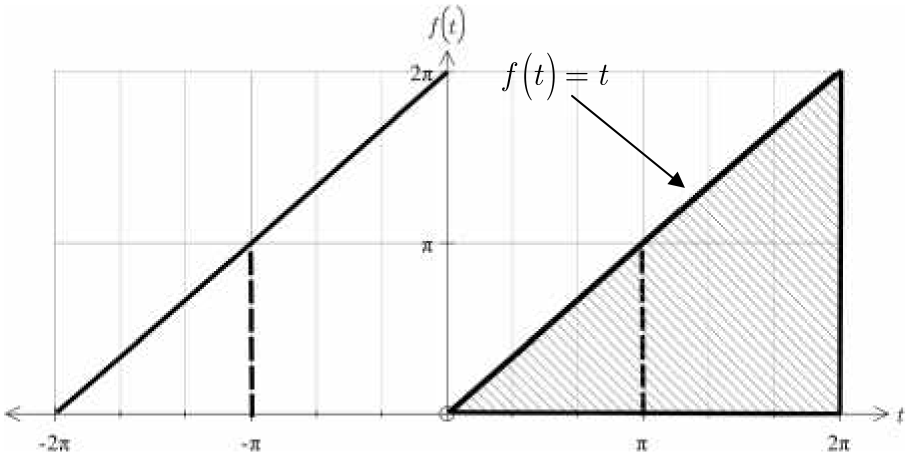

Figure 20

How can we write this area from 0 to 2π in terms of integration?

$$\int_0^{2\pi} \left(t\right)\mathrm{d}t \qquad \left[f\left(t\right) = t \text{ between 0 to } 2\pi\right]$$

Substituting this into (17.3) we have

(17.3)
$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{0}^{2\pi}(t)\,\mathrm{d}t && \left[\text{Replacing } f(t)=t \text{ and changing limits}\right] \\ &= \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} && \left[\text{Integrating}\right] \\ &= \frac{1}{4\pi}\left[(2\pi)^2 - 0^2\right] && \left[\text{Substituting the limits and taking out } 1/2\right] \\ &= \frac{1}{4\pi}\left(4\pi^2\right) = \pi && \left[\text{Cancelling}\right] \end{aligned}$$

Hence the average value of $f(t)$ is $A_0 = \pi$. The value of A_0 is the average value of the function over one period. We have a straight line going for 0 to 2π, so we would expect the average value to be half way between this, so π. This can be a helpful check when evaluating the constant term A_0 of the Fourier series of a function.

How do we find the cosine coefficients A_k ?

We use formula (17.4) and for $f(t)$ we substitute $f(t)=t$ and the limits 0 to 2π. This gives

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(t)\cos(kt)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_{0}^{2\pi}\left[t\cos(kt)\right]\mathrm{d}t && (\dagger) \end{aligned}$$

How do we evaluate the right hand integral $\int_0^{2\pi}\left[t\cos(kt)\right]\mathrm{d}t$?

We need to use integration by parts formula. In our integrand we let $u = t$ and $v' = \cos(kt)$:

$$u = t \qquad\qquad v' = \cos(kt)$$

$$u' = 1 \quad \left[\text{Differentiating}\right] \qquad\qquad v = \int \cos(kt)\,\mathrm{d}t = \frac{\sin(kt)}{k}$$

Substituting these into integration by parts formula, $\int (uv')\,\mathrm{d}t = uv - \int (u'v)\,\mathrm{d}t$, gives

$$\begin{aligned} \int_0^{2\pi}\left[t\cos(kt)\right]\mathrm{d}t &= \left[\frac{t\sin(kt)}{k}\right]_0^{2\pi} - \int_0^{2\pi}\left[\frac{(1)\sin(kt)}{k}\right]\mathrm{d}t && \left[\text{Using } \int (uv')\,\mathrm{d}t = uv - \int (u'v)\,\mathrm{d}t\right] \\ &= \left[\frac{2\pi\sin(2\pi k)}{k} - 0\right] - \left[-\frac{\cos(kt)}{k^2}\right]_0^{2\pi} && \left[\text{Integrating by } \int \sin(kt)\,\mathrm{d}t = -\frac{\cos(kt)}{k}\right] \\ &= \left[\underbrace{0}_{\text{By }(*)} - 0\right] + \frac{1}{k^2}\left[\underbrace{\cos(2\pi k)}_{=1 \text{ by } (**)} - \underbrace{\cos(0)}_{=1}\right] && \left[\text{Substituting limits and using } (*)\ \sin(2\pi k)=0 \quad (**)\ \cos(2\pi k)=1\right] \\ &= \frac{1}{k^2}\left[1-1\right] = 0 \end{aligned}$$

Substituting this $\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t = 0$ into $(\dagger)$ gives

$$A_k = \frac{1}{\pi}\left(0\right) = 0$$

What does $A_k = 0$ mean with respect to the Fourier series we are trying to find?

$A_k = 0$ means that there are *no* cosine terms in the Fourier series.

What else do we need to find?

The values of B_k. (The sine coefficients of the Fourier series). By integrating (17.5) between 0 to 2π rather than $-\pi$ to π respectively gives us

$$\begin{aligned} B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f\left(t\right)\sin\left(kt\right)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad \left[\text{Remember } f\left(t\right) = t \text{ between } 0 \text{ to } 2\pi\right] \\ B_k &= \frac{1}{\pi}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad (\dagger\dagger) \end{aligned}$$

How do we evaluate this integral $\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$?

We need to use integration by parts again to find $\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$:

$$u = t \qquad\qquad v' = \sin\left(kt\right)$$

$$u' = 1 \quad \left[\text{Differentiating}\right] \qquad\qquad v = \int \sin\left(kt\right)\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}$$

Substituting this into integration by parts formula, $\int\left(uv'\right)\mathrm{d}x = uv - \int\left(u'v\right)\mathrm{d}x$, gives

$$\begin{aligned} \int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t &= -\left[\frac{t\cos\left(kt\right)}{k}\right]_0^{2\pi} - \int_0^{2\pi}\left(1\right)\left[\frac{-\cos\left(kt\right)}{k}\right]\mathrm{d}t \\ &= -\frac{1}{k}\left[2\pi\cos\left(2\pi k\right) - 0\right] + \frac{1}{k}\int_0^{2\pi}\cos\left(kt\right)\mathrm{d}t \qquad \left[\begin{array}{l}\text{Substituting limits and}\\ \text{taking out common factor}\end{array}\right] \\ &= -\frac{1}{k}\left[2\pi\underbrace{\cos\left(2\pi k\right)}_{=1 \text{ by } (**)}\right] + \frac{1}{k}\left[\frac{\sin\left(kt\right)}{k}\right]_0^{2\pi} \qquad \left[\text{By } \int\cos\left(kt\right)\mathrm{d}t = \frac{\sin\left(kt\right)}{k}\right] \\ &= -\frac{1}{k}\left[2\pi\right] + \frac{1}{k^2}\left[\underbrace{\sin\left(2\pi k\right)}_{=0 \text{ by } (*)} - \underbrace{\sin\left(0\right)}_{=0}\right] \qquad \left[\text{Substituting limits}\right] \\ &= -\frac{2\pi}{k} \qquad \left[\text{Simplifying}\right] \end{aligned}$$

Substituting this $\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{2\pi}{k}$ into $(\dagger\dagger)$ gives

(*) $\quad \sin\left(2\pi k\right) = 0 \qquad$ (**) $\quad \cos\left(2\pi k\right) = 1 \qquad (\dagger\dagger) \quad B_k = \frac{1}{\pi}\int_0^{2\pi} t\sin\left(kt\right)\mathrm{d}t$

$$B_k = \frac{1}{\not{\pi}}\left(-\frac{2\not{\pi}}{k}\right) = -\frac{2}{k} \qquad [\text{Cancelling}]$$

Putting these values, $A_0 = \pi$, $A_k = 0$ $\left(\text{No cosine terms}\right)$ and $B_k = -\frac{2}{k}$ into the generic Fourier series

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$$

gives

$$f(t) = \pi + \underbrace{0+0+0+\cdots+0}_{\text{No cosine terms}} - 2\sin(t) - \frac{2}{2}\sin(2t) - \frac{2}{3}\sin(3t) - \frac{2}{4}\sin(4t) - \ldots$$

$$= \pi - 2\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4} + \ldots\right] \qquad \left[\begin{array}{l}\text{Taking out}\\ -2\end{array}\right]$$

This is the Fourier series of the sawtooth waveform $f(t)$ as shown in Fig. 19.

Note the long process of finding the Fourier series of a sawtooth waveform. Again it is very easy to make a slip of signs in obtaining the Fourier coefficients. A slip of signs is the *most* common mistake made when evaluating Fourier series.

C2 Sketching Partial Sums of Fourier Series

By using computer algebra system we can sketch $f(t)$ and various partial sums of the Fourier series of $f(t)$ given in the previous example as shown below.

Here is the MAPLE output for 5 terms:

> $f := t \rightarrow \pi - 2\left(\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4}\right)$

$$f := t \rightarrow \pi - 2\sin(t) - \sin(2t) - \frac{2}{3}\sin(3t) - \frac{1}{2}\sin(4t)$$

> $plot(f, -2\pi..2\pi)$

The graph of the first 5 non-zero terms of the Fourier series of the given sawtooth waveform $f(t)$.

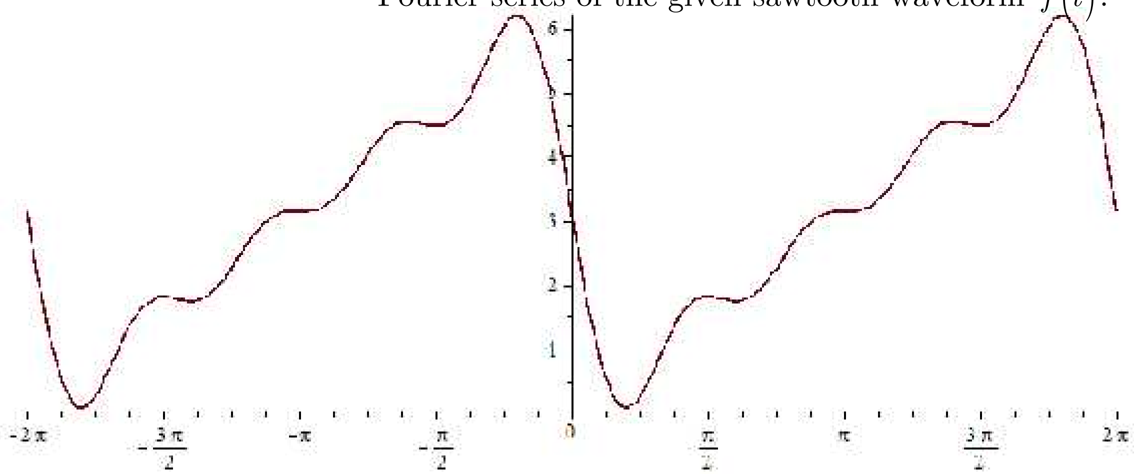

Fig 21

Here is the MAPLE output for 10 terms:

> $g := t \to \pi - 2\left(\sin(t) + \frac{\sin(2\,t)}{2} + \frac{\sin(3\,t)}{3} + \frac{\sin(4\,t)}{4} + \frac{\sin(5\,t)}{5} + \frac{\sin(6\,t)}{6} + \frac{\sin(7\,t)}{7} + \frac{\sin(8\,t)}{8} + \frac{\sin(9\,t)}{9}\right)$

$$g := t \to \pi - 2\sin(t) - \sin(2\,t) - \frac{2}{3}\sin(3\,t) - \frac{1}{2}\sin(4\,t) - \frac{2}{5}\sin(5\,t) - \frac{1}{3}\sin(6\,t) - \frac{2}{7}\sin(7\,t) - \frac{1}{4}\sin(8\,t) - \frac{2}{9}\sin(9\,t)$$

> $plot(g, -2\,\pi\,..2\,\pi)$

The graph of the first 10 non-zero terms of the Fourier series of the given sawtooth waveform $f(t)$.

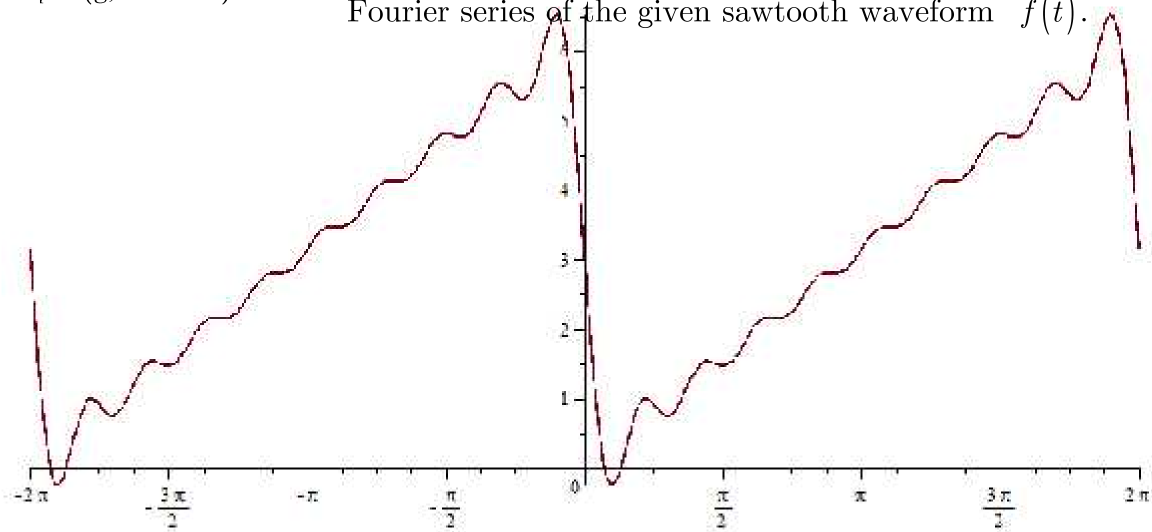

Figure 22

Notice how the shape of the graph is getting closer and closer to the given sawtooth function.

C3 Applications of Fourier Series

The next example is more complicated than the previous Example 5 because it uses multiples of $\frac{\pi}{2}$ in the argument of sine and cosine. However we do *not* need to use ‘integration by parts’ formula to find the Fourier coefficients.

For Example 6 we will need to use the following trigonometric result:

($) $$\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 1 & \text{if } k = 1,\ 5,\ 9, \ldots \\ 0 & \text{if } k = \text{even} \\ -1 & \text{if } k = 3,\ 7,\ 11, \ldots \end{cases}$$

You can see this result from the following graph:

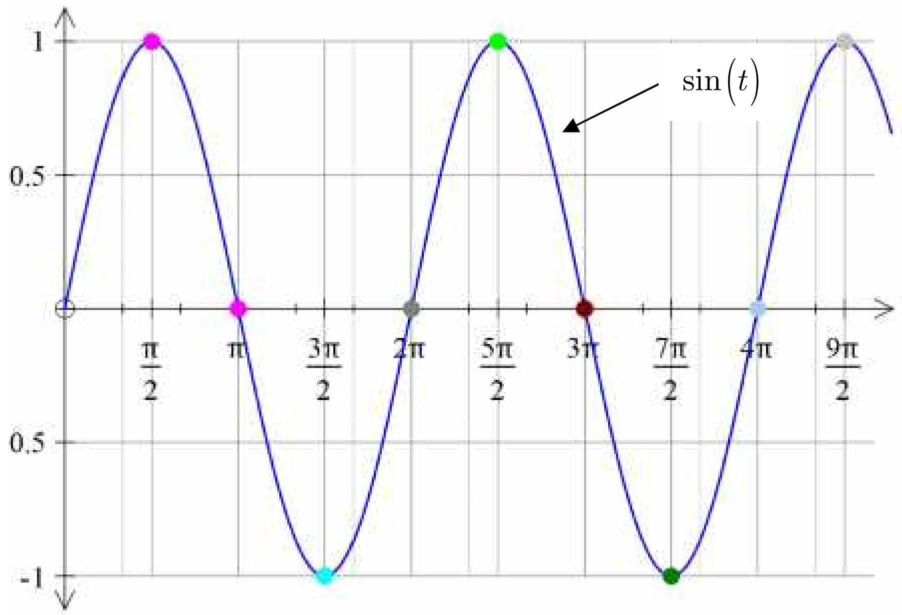

Figure 23

We also have

($$) $$\cos\left(k\frac{\pi}{2}\right)=\begin{cases}-1 & \text{if } k=2,\ 6,\ 10,\dots\\ 0 & \text{if } k=\text{odd}\\ 1 & \text{if } k=4,\ 8,\ 12,\dots\end{cases}$$

Again we can see this result from the following graph:

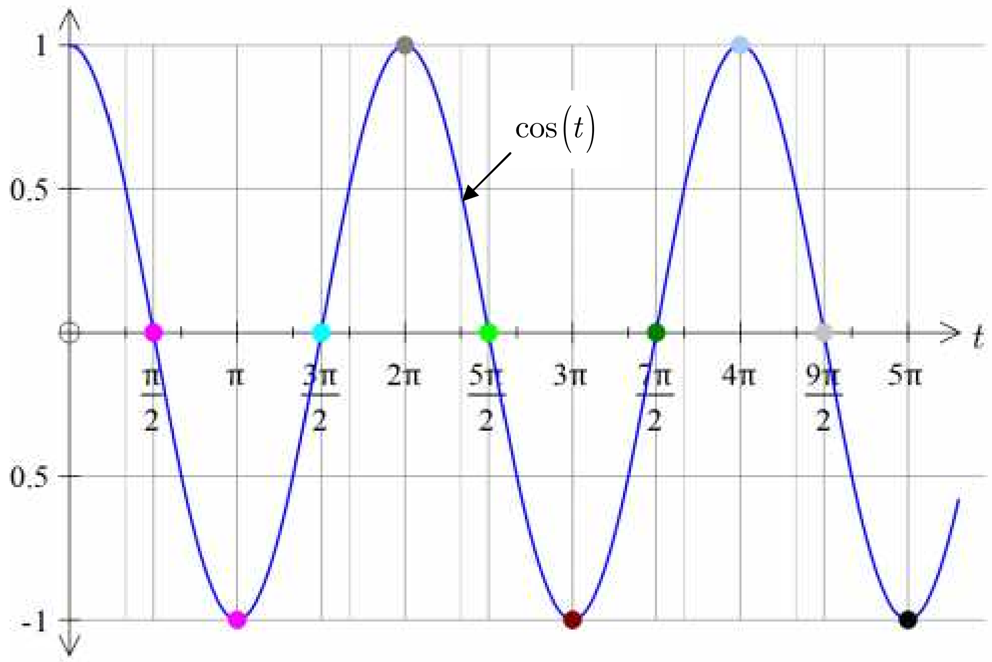

Figure 24

These results are *not* well known but you should be able to derive them by examining the graphs of sine and cosine functions shown above. We will assume these results and use them in Example 6.

Example 6 (Mechanical)

A pulse force $f(t)$ is applied to a mechanical system which has the following graph:

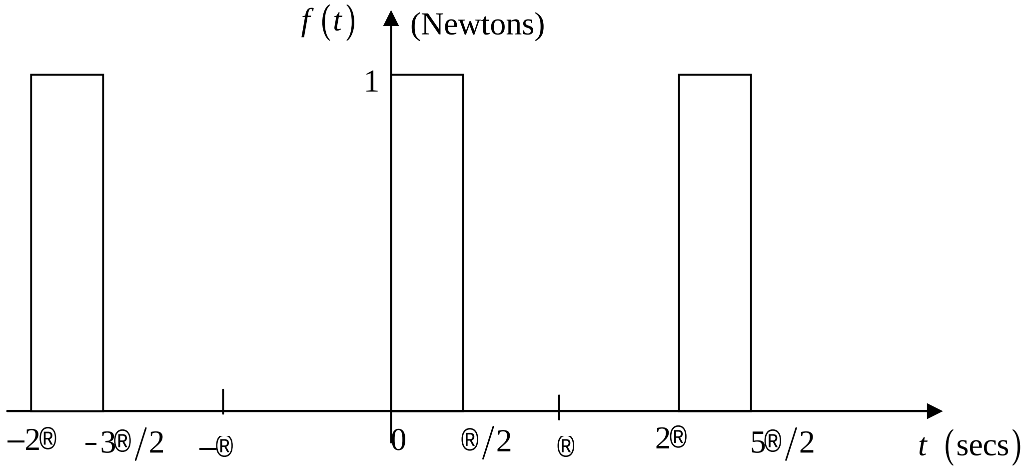

Figure 25

Determine the Fourier series of $f(t)$.

<u>Solution</u>

We need to find all the sine, cosine and constant coefficients in the Fourier series.

How do we find these?

By using the given formulae. For the constant term (average value) A_0 we use

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

What is $f(t)$ equal to between $-\pi$ and π ?

From the graph in Fig. 25 we can see that $f(t)=0$ between $-\pi$ to 0 and $\pi/2$ to π but $f(t)=1$ between 0 to $\pi/2$. Hence we only need to use $f(t)=1$ with limits 0 and $\pi/2$ because the other values are zero.

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(0)\,\mathrm{d}t + \int_{0}^{\pi/2}(1)\,\mathrm{d}t + \int_{\pi/2}^{\pi}(0)\,\mathrm{d}t\right] \\
&= \frac{1}{2\pi}\int_{0}^{\pi/2}(1)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left[t\right]_0^{\pi/2} && \left[\text{Integrating}\right] \\
&= \frac{1}{2\pi}\left[\frac{\pi}{2}-0\right] && \left[\text{Substituting limits}\right] \\
&= \frac{1}{2\not\pi}\,\frac{\not\pi}{2} = \frac{1}{4}
\end{aligned}$$

The average value of this function over a complete period is $A_0 = 1/4$.

What else do we need to find?

The cosine coefficients A_k by formula (17.4):

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f\left(t\right)\cos\left(kt\right)\right]\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{0}^{\pi/2}\left[\left(1\right)\cos\left(kt\right)\right]\,\mathrm{d}t && \left[\text{Using } f\left(t\right)=1 \text{ between the limits 0 to } \pi/2\right] \\
&= \frac{1}{\pi}\left[\frac{\sin\left(kt\right)}{k}\right]_0^{\pi/2} && \left[\text{Integrating } \cos\left(kt\right)\right] \\
&= \frac{1}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right)-0\right] && \left[\text{Taking out a common factor of } 1/k \text{ and substituting limits}\right]
\end{aligned}$$

What is $\sin\left(k\frac{\pi}{2}\right)$ *equal to?*

We use the above stated result:

$$(\$)\qquad \sin\left(k\frac{\pi}{2}\right)=\begin{cases}1 & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -1 & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

How do we find the values for $A_k=\frac{1}{k\pi}\sin\left(k\frac{\pi}{2}\right)$*?*

By multiplying this ($) by $1/k\pi$:

$$A_k=\frac{1}{k\pi}\sin\left(k\frac{\pi}{2}\right)=\begin{cases}1/k\pi & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -1/k\pi & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

What does this mean?

There are *no even* cosine terms in the Fourier series. If $k=1,\ 5,\ 9,\ldots$ then the cosine coefficient $A_k=\frac{1}{k\pi}$ and we have

$$(*)\qquad A_1=\frac{1}{\pi},\ A_5=\frac{1}{5\pi},\ A_9=\frac{1}{9\pi},\ldots$$

If $k=3,\ 7,\ 11,\ldots$ then the cosine coefficient is negative, $A_k=-\frac{1}{k\pi}$, and we have

$$(**)\qquad A_3=-\frac{1}{3\pi},\ A_7=-\frac{1}{7\pi},\ A_{11}=-\frac{1}{11\pi},\ldots$$

What is our next step?

We have to find the sine coefficients, B_k, of the Fourier series. This is given by (17.5):

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} \left[(1)\sin(kt)\right]\mathrm{d}t \qquad \left[\text{Using } f(t)=1 \text{ between the limits } 0 \text{ to } \pi/2\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(kt)}{k}\right]_0^{\pi/2} \qquad \left[\text{Integrating } \sin(kt)\right]$$

$$= -\frac{1}{k\pi}\left[\cos\left(k\frac{\pi}{2}\right) - \cos(0)\right] \qquad \left[\text{Taking out a common factor of } -1/k \text{ and substituting limits}\right]$$

$$= -\frac{1}{k\pi}\left[\cos\left(k\frac{\pi}{2}\right) - 1\right]$$

We use the above stated result:

$$(\$\$) \qquad \cos\left(k\frac{\pi}{2}\right) = \begin{cases} -1 & \text{if } k = 2,\ 6,\ 10, \ldots \\ 0 & \text{if } k = \text{odd} \\ 1 & \text{if } k = 4,\ 8,\ 12, \ldots \end{cases}$$

And substitute these values of $\cos\left(k\frac{\pi}{2}\right)$ into the last line of the above derivation:

$$B_k = -\frac{1}{k\pi}\left[\cos\left(k\frac{\pi}{2}\right) - 1\right]$$

What is the value of B_k *if* $k = 2,\ 6,\ 10, \ldots$ *?*

$$(\dagger) \qquad B_k = -\frac{1}{k\pi}\left[-1-1\right] = \frac{2}{k\pi} \qquad \left[k = 2,\ 6,\ 10, \cdots\right]$$

What is the value of B_k *if* k *is odd?*

$$(\dagger\dagger) \qquad B_k = -\frac{1}{k\pi}\left[0-1\right] = \frac{1}{k\pi} \qquad \left[k \text{ is odd } 1,\ 3,\ 5,\ \cdots\right]$$

What is the value of B_k *if* $k = 4,\ 8,\ 12, \ldots$ *?*

$$B_k = -\frac{1}{k\pi}\left[1-1\right] = 0 \qquad \left[k = 4,\ 8,\ 12, \cdots\right]$$

Hence the non-zero values of the sine coefficients, B_k, in the Fourier series is when $k = 2,\ 6,\ 10, \ldots$ and when k is odd. Substituting these values of k into $(\dagger)$ and $(\dagger\dagger)$ gives

$$B_1 = \frac{1}{\pi},\ B_2 = \frac{2}{2\pi},\ B_3 = \frac{1}{3\pi},\ B_4 = 0,\ B_5 = \frac{1}{5\pi},\ B_6 = \frac{2}{6\pi},\ B_7 = \frac{1}{7\pi},\ \ldots$$

How do we obtain the Fourier series?

Substitute the coefficients A_k and B_k into the generic Fourier series.

We substitute the constant term $A_0 = \frac{1}{4}$. *What do we substitute for the cosine coefficients* A_k *?*

Using (*) and (**) we have

$$A_1 = \frac{1}{\pi},\ A_3 = -\frac{1}{3\pi},\ A_5 = \frac{1}{5\pi},\ A_7 = -\frac{1}{7\pi},\ \ldots$$

Remember there are *no even* cosine coefficients because $A_k = 0$ when $k = \text{even}$.

Putting these and the above B_k values into

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$$

gives

$$f(t) = \frac{1}{4} + \left[\underbrace{\frac{\cos(t)}{\pi} + 0 - \frac{\cos(3t)}{3\pi} + 0 + \frac{\cos(5t)}{5\pi} - \ldots}_{\text{Cosine terms}}\right]$$
$$+ \left[\underbrace{\frac{\sin(t)}{\pi} + \frac{2\sin(2t)}{2\pi} + \frac{\sin(3t)}{3\pi} + 0 + \frac{\sin(5t)}{5\pi} + \frac{2\sin(6t)}{6\pi} + \ldots}_{\text{Sine terms}}\right]$$
$$= \frac{1}{4} + \frac{1}{\pi}\left[\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \cdots + \sin(t) + \sin(2t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + ..\right] \qquad \left[\text{Taking out } \frac{1}{\pi}\right]$$

This is the Fourier series of the pulse force $f(t)$ shown in Fig. 25.

Even though we don't apply the 'integration by parts' formula for the rectangular waveform of Example 6 it is still a long process to obtain the Fourier series. There are so many chances of making a mistake in such a long calculation.

SUMMARY

The integration used to find the Fourier coefficients generally entails the method of 'integration by parts'.

If we have a waveform which involves fractions of π then we need to be careful in covering all possible values of k in evaluating the Fourier coefficients.

(*) $A_1 = \frac{1}{\pi},\ A_5 = \frac{1}{5\pi},\ A_9 = \frac{1}{9\pi}$ (**) $A_3 = -\frac{1}{3\pi},\ A_7 = -\frac{1}{7\pi},\ A_{11} = -\frac{1}{11\pi}$