SECTION B **Approximations of Irrational Numbers**

By the end of this section you will be able to

- find convergents of continued fractions
- convert an irrational number into a continued fraction

In this section we examine rational approximations to irrational numbers by using continued fractions. For example

$$® = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3, \cdots]$$

$$e = [2;\ 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \cdots]$$

$$\sqrt{2} = [1;\ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,\ \cdots]$$

$$¤ = \frac{1+\sqrt{5}}{2} = [1;\ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,\ \cdots]$$

We can chop off an infinite continued fraction after a finite number of steps which will then give us a rational approximation to the irrational number.
For example if we chop off the continued fraction of ® after the first two terms:

$$® \simeq [3;\ 7] = \frac{22}{7}$$

If we chop off the fraction after the first four terms we have

$$® \simeq [3;\ 7,\ 15,\ 1] = \frac{355}{113}$$

We will find in this section that these are good approximations to ® .
The more terms we consider the better our approximation.

B1 Convergents

In this section we examine cutting or chopping off the continued fraction at various points.
The continued fraction of $\frac{59}{13} = [4;\ 1,\ 1,\ 6]$. We can use this continued fraction to approximate $\frac{59}{13}$. For example the first approximation is 4 which is the first term in the continued fraction. Other approximations are

$$[4;\ 1] = 4 + \frac{1}{1} = 5$$

$$[4;\ 1,\ 1] = 4 + \frac{1}{1+1} = \frac{9}{2}$$

$$[4;\ 1,\ 1,\ 6] = 4 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6}}} = \frac{59}{13}$$

Except for the last approximation all the others are less than or greater than $\frac{59}{13}$. Each time it is *closer* to the actual value. The graphical display is shown next:

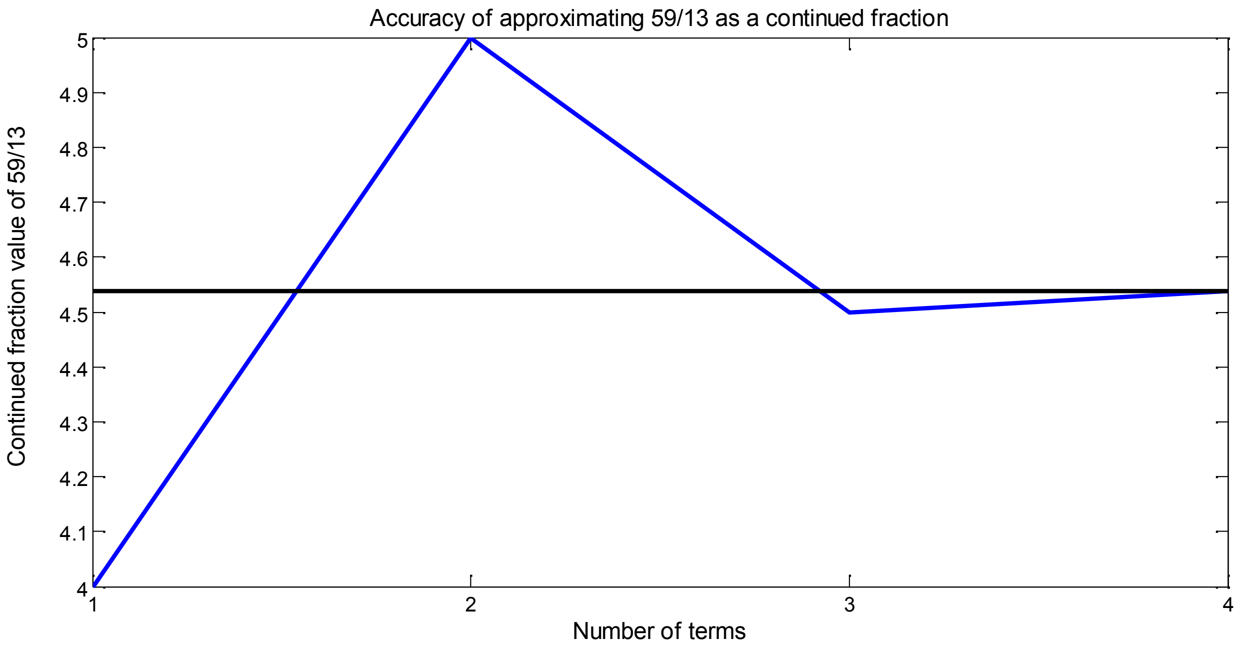


Figure 1
The figure above shows how increasing the number of terms in the continued fraction expansion changes the approximation of $\frac{59}{13}$, the black solid line being the actual value of $59/13$. We can see that every time we increase the number of terms in our continued fraction the approximation becomes closer to the actual value.
Cutting off the continued fraction at a particular stage is called a *convergent*.

Definition (15.3).
The continued fraction made from $[a_1;\ a_2,\ a_3,\ \cdots,\ a_k]$ by cutting off the expansion after the kth partial denominator a_k where $k \le n$ is called the kth **convergent** of the given continued fraction. This kth convergent is normally denoted by C_k.

We can express each convergent in terms of ratios $\frac{p_k}{q_k}$, that is $C_k = \frac{p_k}{q_k}$. For the above example we have:

k	$[a_1;\ a_2,\ a_3,\ \cdots,\ a_k]$	$C_k = \frac{p_k}{q_k}$ (Rational Approx)
1	[4]	$\frac{4}{1}$
2	[4; 1]	$\frac{5}{1}$
3	[4; 1, 1]	$\frac{9}{2}$
4	[4; 1, 1, 6]	$\frac{59}{13}$ (Exact)

TABLE 1

Can we find the last convergent $\frac{59}{13}$ *from the first three convergents or do we have to evaluate this from the beginning by finding the continued fraction?*

We can evaluate a rational approximate directly rather than work out the continued fraction each time.

We want to find a relationship between $[a_1;\ a_2,\ a_3,\ \cdots,\ a_k]$ and $\frac{p_k}{q_k}$.

Let us create a table of the given continued fraction $[4;\ 1,\ 1,\ 6]$ with the numerator and denominator of the convergents at each stage:

k	a_k	p_k	q_k
1	4	4	1
2	1	5	1
3	1	9	2
4	6	59	13

TABLE 2

Can you spot a pattern between p_k *and* q_k *in the table?*

Can we find p_k *and* q_k *from the previous values in the table?*

Any p_k term can be found by the previous two terms, p_{k-1}, p_{k-2} and the a_k term.

For example $p_4 = 59$ can be found by:

$$59 = (6\times 9)+5 \text{ (These numbers are highlighted in the table)}$$

Another example $p_3 = 9$:

$$9 = (1\times 5)+4$$

Similarly $q_4 = 13$ and observe

$$13 = (6\times 2)+1$$

Also $q_3 = 2$:

$$2 = (1\times 1)+1$$

We can write the general formulae as follows:

(15.4) $$p_k = a_k p_{k-1} + p_{k-2}$$

(15.5) $$q_k = a_k q_{k-1} + q_{k-2}$$

Note that when $k = 1$ the last terms in the formulae are $p_{-1},\ q_{-1}$ and when $k = 2$ we have p_0 and q_0. *What are these values?*

To make the arithmetic easier we start off with $p_{-1} = 0,\ q_{-1} = 1$ and

$$p_0 = 1 \text{ and } q_0 = 0$$

Choosing different values will *not* give the simplest fraction.

Example 6

Given the continued fraction for $\frac{19}{51} = [0;\ 2,\ 1,\ 2,\ 6]$, find the convergents $C_1,\ C_2,\ \cdots,\ C_5$.

<u>Solution</u>

k	a_k	p_k	q_k
−1		0	1
0		1	0

1	0	$(0\times1)+0=0$	$(0\times0)+1=1$
2	2	$(2\times0)+1=1$	$(2\times1)+0=2$
3	1	$(1\times1)+0=1$	$(1\times2)+1=3$
4	2	$(2\times1)+1=3$	$(2\times3)+2=8$
5	6	$(6\times3)+1=19$	$(6\times8)+3=51$

Reading off the values in the last two columns gives the convergents:

$$C_1=\frac{p_1}{q_1}=\frac{0}{1}=0,\; C_2=\frac{p_2}{q_2}=\frac{1}{2},\; C_3=\frac{p_3}{q_3}=\frac{1}{3},\; C_4=\frac{p_4}{q_4}=\frac{3}{8} \text{ and } C_5=\frac{p_5}{q_5}=\frac{19}{51}$$

Note that at each stage we get better and better rational approximations to the given fraction $\frac{19}{51}$.

Example 7

Determine the rational number of the following simple continued fraction $[2;\ 3,\ 1,\ 4,\ 2]$.

Solution

Similarly to the above example we have

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	2	$(2\times1)+0=2$	$(2\times0)+1=1$
2	3	$(3\times2)+1=7$	$(3\times1)+0=3$
3	1	$(1\times7)+2=9$	$(1\times3)+1=4$
4	4	$(4\times9)+7=43$	$(4\times4)+3=19$
5	2	$(2\times43)+9=95$	$(2\times19)+4=42$

Hence the rational number equal to $[2;\ 3,\ 1,\ 4,\ 2]$ is $\frac{95}{42}$.

B2 Continued Fractions of Irrational Numbers

One of the main application of continued fractions is to approximate irrational numbers.
What is an irrational number?

A number which *cannot* be written as a fraction of two integers. For example ®, $\sqrt{2}$ and e are *all* examples of irrational numbers.

Since we *cannot* write an irrational number as a fraction of two integers so the continued fraction of an irrational number will continue forever.

First we find some of the terms of a continued fraction for ® .

Example 8

Determine the first five terms of the continued fraction for ® $=3.141592653\ldots$

Solution

The procedure is similar to finding the continued fraction for a rational number described in section 15A. However in this case we need to use the floor function.

Definition (2.7) of chapter 2:

The floor function is denoted by $\lfloor x \rfloor$ and is the greatest integer *less than or equal* to *x*.

Step 1 First determine $\lfloor ® \rfloor = \lfloor 3.141592653\cdots \rfloor = 3$. This means that

$$® = 3 + 0.141592653\cdots$$

Writing ® as the whole number 3 plus a fractional part, $0.141592653\cdots$;

$$® = 3 + 0.141592653\cdots$$

Step 2 Next we write

$$® = 3 + 0.141592653\cdots = 3 + \cfrac{1}{\cfrac{1}{0.141592653\cdots}} = 3 + \frac{1}{7.062513335\cdots}$$

Step 3 This time we repeat Steps 1 and 2.
Find the floor function of $7.062513335\cdots$:

$$\lfloor 7.062513335\cdots \rfloor = 7$$

We have

$$® = 3 + \frac{1}{7.062513335\ldots} = 3 + \frac{1}{7 + 0.062513335\ldots} = 3 + \cfrac{1}{7 + \cfrac{1}{\cfrac{1}{0.062513335\ldots}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15.99658697...}}$$

Repeating this process we have

$$® = 3 + \cfrac{1}{7 + \cfrac{1}{15.99658697\cdots}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + 0.99658697\cdots}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{\cfrac{1}{0.99658697\cdots}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1.00342472\cdots}}}$$

Also

$$® = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1.00342472\ldots}}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 0.00342472\ldots}}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{0.00342472\ldots}}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292.0000035\ldots}}}}$$

The first five terms of the continued fraction for ® is $[3;\ 7,\ 15,\ 1,\ 292]$.

These continued fractions provide very good rational approximations to ® .

In general, when we approximate an irrational number using continued fractions the accuracy will increase as we increase the number of terms. If we look at the numbers gained in example 8 we can see this. We can achieve further insight by looking at graphs as well. We see from the graph that at the first approximation, that is $[3] = 3$, our approximation is

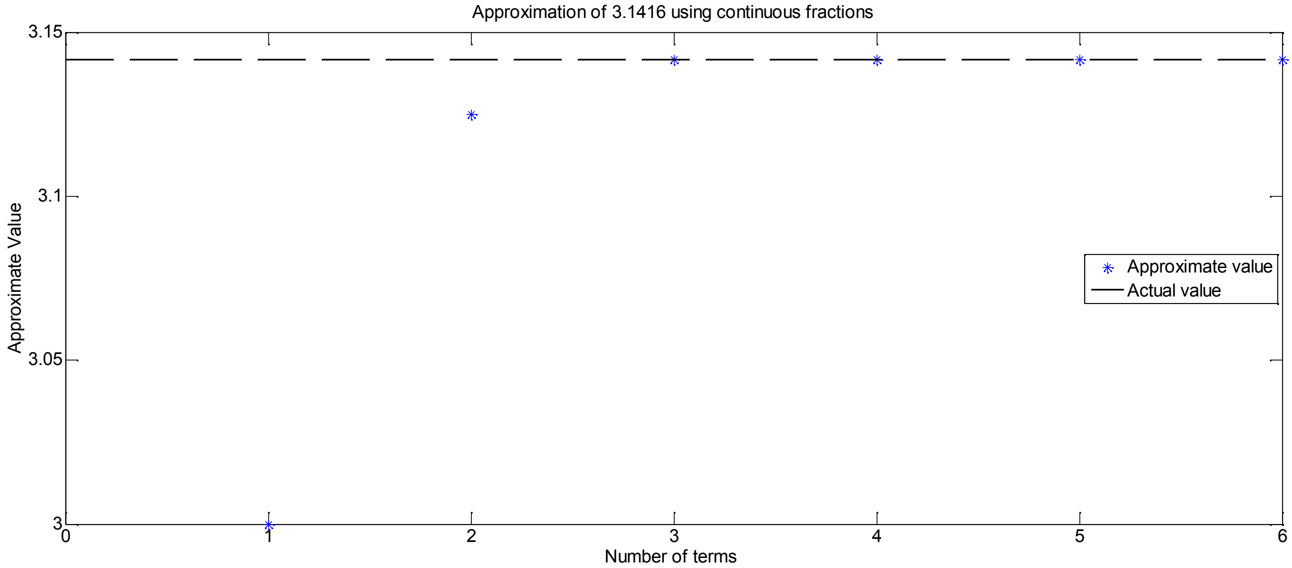


Figure 2

way off since it is the floor function of π. The next approximation is given by $[3;7] = 3 + \frac{1}{7} = \frac{22}{7}$, this fraction was commonly used as an approximation for π before calculators were common place. In decimal form, it has the value 3.142857 so it is indeed a good approximation. However, if we look at the graph, we can see it is still off from the *exact* value. We can see graphically that after the 3rd convergent the approximations are really rather excellent. If we were to zoom in on the graph in that region, we'd see the figure below.

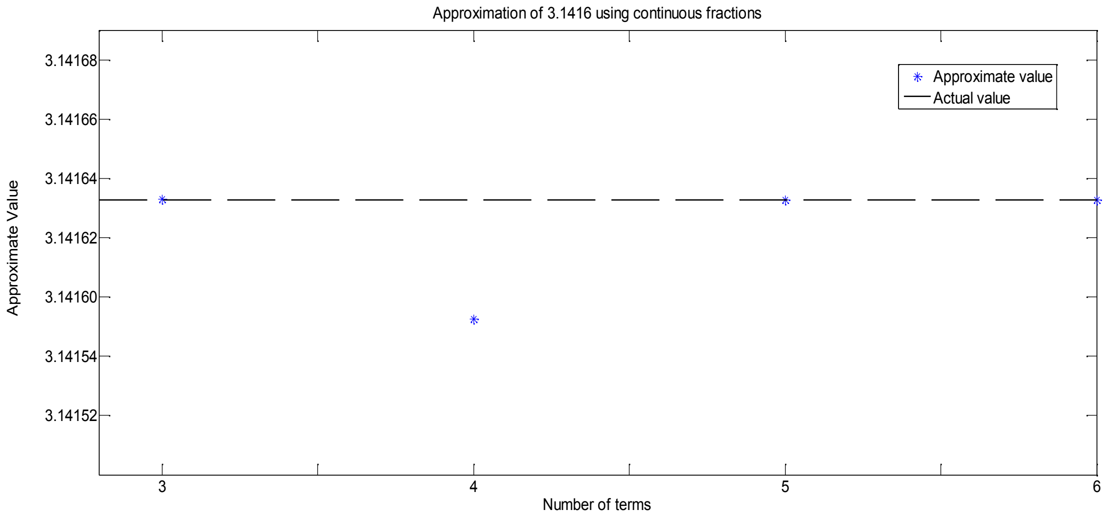


Figure 3

We can now see that there is still some oscillatory behaviour in our approximation. But the errors are getting increasingly small. Indeed, for 6th convergent, out approximation for π would be $[3;\ 7\ 15\ 1\ 291\ 1] = 3.14159265346737$ which is $\pi + 1.22356 \times 10^{-10}$, so it is indeed an excellent approximation.

We use the notation $\sim_n$ to represent the final term of the continued fraction expansion. Of course $\sim_n$ will have a non-stopping decimal expansion. This means that for the above example $\sim_1 = ®$. Let the first entry of the continued fraction $a_1 = \lfloor \sim_1 \rfloor$. Then ® is equal to a_1 plus the fractional part which is $\sim_1 - a_1$. Now $\sim_2$ is the reciprocal of this fractional part:

$$\sim_2 = \frac{1}{\sim_1 - a_1}$$

Then a_2 is the floor function of this $a_2 = \lfloor \sim_2 \rfloor$.

Summarizing the above Example 8 and this $\sim_n$ notation we have:

From Step 1:

$$\sim_1 = ® = 3.141592653\ldots \qquad a_1 = \lfloor \sim_1 \rfloor = \lfloor 3.141592653\ldots \rfloor = 3$$

By Step 2 we have:

$$® - 3 = 0.141592653\cdots$$

$$\sim_2 = \frac{1}{\sim_1 - a_1} = \frac{1}{® - 3} = \frac{1}{0.141592653\cdots} = 7.062513335\ldots$$

The floor function of this is $a_2 = \lfloor 7.062513335\ldots \rfloor = 7$.

Similarly $\sim_3 = \dfrac{1}{\sim_2 - a_2} = 15.99658697...$ and $a_3 = \lfloor \sim_3 \rfloor = \lfloor 15.99658697... \rfloor = 15$.

We carry on this manner and obtain the infinite continued fraction $[a_1;\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots]$ for ®. That is $® = [3;\ 7,\ 15,\ 1,\ 292,\ \cdots]$.

Definition (15.6).

Any positive *irrational* number $\sim$ has an *infinite* continued fraction expansion

$$\sim = [a_1;\ a_2,\ a_3,\ a_4,\ \cdots]$$

where

$$\sim_1 = \sim \text{ (given irrational number)}$$

$$a_n = \lfloor \sim_n \rfloor \qquad \text{[Floor function]}$$

$$\sim_{n+1} = \frac{1}{\sim_n - a_n}$$

Example 9

Determine the continued fraction expansion of $\sqrt{2} = 1.414213562\ldots$

<u>Solution</u>

Using the procedure outlined above and Definition (15.6) we have $\sim_1 = 1.414213562\ldots$:

$$a_1 = \lfloor \sim_1 \rfloor = \lfloor 1.414213562\ldots \rfloor = 1$$

Substituting $n = 1$ into $\sim_{n+1} = \dfrac{1}{\sim_n - a_n}$ gives $\sim_{1+1} = \dfrac{1}{\sim_1 - a_1}$. Putting $a_1 = 1$ and $\sim_1 = 1.414213562\ldots$ into this yields:

$$\sim_2 = \frac{1}{\sim_1 - a_1} = \frac{1}{1.414213562\ldots - 1} = \frac{1}{0.414213562\ldots} = 2.414213562\cdots$$

Hence $a_2 = \lfloor \sim_2 \rfloor = \lfloor 2.414213562\cdots \rfloor = 2$.

Repeating this process we have

$$\sim_3 = \frac{1}{\sim_2 - a_2} = \frac{1}{2.414213562\ldots - 2} = \frac{1}{0.414213562\ldots} = 2.414213562\cdots$$

Note that $\sim_3 = \sim_2$. We are going to use the same formula therefore we will get the same result each time, that is

$$\sim_4 = \sim_3 = \sim_2 = 2.414213562\cdots$$

Repeating this process we have

$$\sim_6 = \sim_5 = \sim_4 = \sim_3 = \sim_2$$

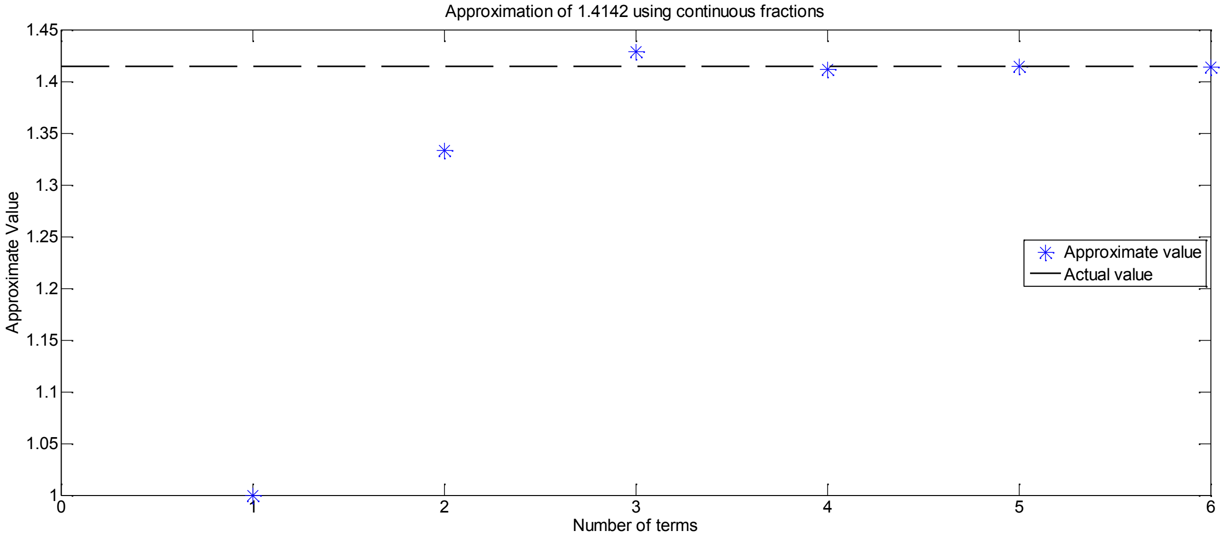
This means that $a_6 = a_5 = a_4 = a_3 = a_2 = \lfloor 2.414213562\cdots \rfloor = 2$. Hence the continued fraction expansion of

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \cdots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}}}$$

We can write this in a more compact form by placing a bar over the 2 to indicate that this integer is being repeated as follows:

$$\sqrt{2} = \left[1;\ \overline{2}\right] \text{ or } \sqrt{2} = \left[1;\ \langle 2 \rangle\right]$$

Again, we can see the behaviour of our approximation to $\sqrt{2}$ using a graph



Where it becomes clear that we have a good approximation after the 6^{th} convergence. It takes longer to reach a stable approximation, unlike what we saw for π.

Example 10

Determine the continued fraction expansion of $\sqrt{6} = 2.449489743\ldots$

<u>Solution</u>

We use the above process but this time we will place the values in tabular form. It is a lot easier to use the x^{-1} button on your calculator.

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$ (Floor function)
1	$\sim_1 = \sqrt{6}$	2.449489743…	2
2	$\sim_2 = \dfrac{1}{\sim_1 - a_1} = \dfrac{1}{2.449489743\ldots - 2}$	2.224744871…	2
3	$\sim_3 = \dfrac{1}{\sim_2 - a_2} = \dfrac{1}{2.224744871\ldots - 2}$	4.449489743…	4
4	$\sim_4 = \dfrac{1}{\sim_3 - a_3} = \dfrac{1}{4.449489743\ldots - 4}$	2.224744871…	2
5	$\sim_5 = \dfrac{1}{\sim_4 - a_4} = \dfrac{1}{2.224744871\ldots - 2}$	4.449489743…	4
6	$\sim_6 = \dfrac{1}{\sim_5 - a_5} = \dfrac{1}{4.449489743\ldots - 4}$	2.224744871…	2
⋮	⋮	⋮	⋮

You will see this pattern of

$$\sim_2 = \sim_4 = \sim_6 = \cdots = 2 \text{ and } \sim_3 = \sim_5 = \sim_7 = \cdots = 4$$

repeated.

Hence we can write the continued fraction of $\sqrt{6}$ as $\left[2;\ \langle 2,\ 4\rangle\right]$.

Example 11

Find the irrational number given by $\sim = \left[\langle 3,\ 2\rangle\right]$.

Solution

What does this notation $\sim = \left[\langle 3,\ 2\rangle\right]$ *mean?*

It is the continued fraction given by

$$\sim = 3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \ddots}}}}}}$$

The part of the expression which is blue is exactly the original continued fraction. Hence the blue expression is equal to $\sim$. The above becomes

$$\sim = 3 + \cfrac{1}{2 + \cfrac{1}{\sim}}$$

Multiplying the last term on the Right Hand Side by $\dfrac{\sim}{\sim}(=1)$ gives

$$\sim = 3 + \frac{\sim}{2\sim + 1}$$

Multiplying both sides by $2\sim + 1$ yields

$$\sim(2\sim+1)=3(2\sim+1)+\sim$$

$$2\sim^2+\sim=6\sim+3+\sim \qquad [\text{Expanding}]$$

$$2\sim^2-6\sim-3=0$$

Using the quadratic formula to solve $2\sim^2-6\sim-3=0$ with $a=2,\ b=-6$ and $c=-3$:

$$\sim=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{6\pm\sqrt{(-6)^2-[4\times2\times(-3)]}}{2\times2}$$

$$=\frac{6\pm\sqrt{60}}{4}$$

$$=\frac{6\pm\sqrt{4\times15}}{4}=\frac{3\pm\sqrt{15}}{2}$$

$\sim$ *cannot* be negative so our only solution is

$$\sim=\frac{3+\sqrt{15}}{2}\approx3.436491673$$

Example 12

Find the real number given by $\sim=[5,\ \langle4,\ 2\rangle]$.

Solution

The notation $\sim=[5,\ \langle4,\ 2\rangle]$ means

$$\sim=5+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\ddots}}}}}}$$

This time $\sim$ is *not* a copy of itself because we have a 5 at the start which ruins the pattern of 2 and 4's. *How can we find $\sim$ in this case?*

We can write the repeated continued fraction as another symbol like . This means we have

$$\sim=5+\frac{1}{\ } \qquad (*)$$

If we can find then we can determine $\sim$. *How can we find ?*

This is a repeated continued fraction:

$$=4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\ddots}}}}}}$$

Again the continued fraction is copied out again as shown in blue. We have

$$= 4 + \cfrac{1}{2 + \cfrac{1}{\ }}$$

Repeating the same algebraic technique as in the above example we have

$$= 4 + \frac{\ }{2\ \ +1} \qquad \left[\text{Multiplying the last term by } \frac{\ }{\ } = 1\right]$$

$$(2\ \ +1) = 4(2\ \ +1) + \qquad [\text{Multiplying both sides by } 2\ \ +1]$$

$$2\ \ ^2 + \ \ = 8\ \ + 4 +$$

$$2\ \ ^2 - 8\ \ - 4 = 0$$

$$\ ^2 - 4\ \ - 2 = 0 \qquad [\text{Dividing both sides by } 2]$$

Using the quadratic formula with $a = 1$, $b = -4$ and $c = -2$ gives

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - [4 \times 1 \times (-2)]}}{2}$$

$$= \frac{4 \pm \sqrt{24}}{2} = \frac{4 \pm \sqrt{4 \times 6}}{2} = \frac{4 \pm 2\sqrt{6}}{2} = 2 \pm \sqrt{6}$$

We ignore the negative solution so we have $\ \ = 2 + \sqrt{6}$. Substituting this (*) yields

$$\sim\ = 5 + \frac{1}{\ } = 5 + \frac{1}{2 + \sqrt{6}}$$

Rationalizing the denominator of this by multiplying the last term on the right hand side by $\frac{2 - \sqrt{6}}{2 - \sqrt{6}} (= 1)$:

$$\sim\ = 5 + \frac{1}{2 + \sqrt{6}} \frac{2 - \sqrt{6}}{2 - \sqrt{6}} = 5 + \frac{2 - \sqrt{6}}{4 - 6}$$

$$= 5 + \frac{\sqrt{6} - 2}{2} = \frac{10 + \sqrt{6} - 2}{2} = \frac{8 + \sqrt{6}}{2}$$

The real numbers determined in the last two examples, $\frac{3 + \sqrt{15}}{2}$ and $\frac{8 + \sqrt{6}}{2}$, have a special name, *quadratic irrational***.** The formal definition of a quadratic irrational is:

Definition (15.7).

Any number $\sim$ of the form $\sim\ = \frac{a + b\sqrt{c}}{d}$ where $d > 0$ and $c > 0$ is called a **quadratic irrational**.

It can be shown that a number is a positive quadratic irrational $\Leftrightarrow$ it has a periodic simple continued fraction.

We can summarize and classify type of real number and its continued fractions.

Type of real number	**Type of continued fractions**
Non-negative rational numbers	Finite simple continued fractions
Positive quadratic irrationals	Periodic simple continued fractions
All other positive real numbers	Infinite aperiodic simple continued fractions

SUMMARY
We can approximate irrational numbers by using continued fractions.