Exercise **15(c)**

1. Determine the first 5 convergents of the following numbers and the difference between the actual value and your 5th convergent correct to 2sf.
   (a) $\sqrt{2} = [1;\ \langle 2 \rangle]$ (b) $\sqrt{5} = [2;\ \langle 4 \rangle]$ (c) $\sqrt{3} = [1;\ \langle 1,\ 2 \rangle]$
   Which of these numbers has the largest error with the 5th convergent?

2. (i) Determine the simple continued fraction of $\frac{43}{5}$.
   (ii) Find the general solution of the Diophantine equation $43x + 5y = 7$.

3. (i) Determine the simple continued fraction of $\frac{106}{19}$.
   (ii) Obtain the general solution of the linear Diophantine equation $106x + 19y = 100$.

4. By using continued fractions find the general solution of the following Diophantine equations:
   (a) $86x + 17y = 3$ (b) $111x + 23y = 5$ (c) $201x + 51y = 9$

5. Find the first convergent of the following irrational numbers that is within 0.0001 of its actual value:
   (a) $\sqrt{10} = [3;\ \langle 6 \rangle]$ (b) $\sqrt{15} = [3;\ \langle 1,\ 6 \rangle]$ (c) $\frac{1+\sqrt{10}}{3} = [1;\ \langle 2,\ 1,\ 1 \rangle]$

6. Prove the following result:
   If q_k is the denominator of the *k*th convergent of $[a_0;\ a_1,\ a_2,\ \cdots,\ a_n]$ then
   $$q_{k-1} \le q_k \text{ for } 1 \le k \le n$$

7. Show that if $\frac{p_n}{q_n}$ is the *n*th convergent to the irrational number $\sim$ then
   $$\left| \sim - \frac{p_n}{q_n} \right| < \frac{1}{{q_n}^2}$$

8. Prove that an infinite continued fraction is an irrational number.

9. Prove the following result:
   If the infinite continued fractions $[a_0;\ a_1,\ a_2,\ \cdots]$ and $[b_0;\ b_1,\ b_2,\ \cdots]$ are equal, then $a_n = b_n$ for all $n \ge 0$.

Brief Solutions

1. (a) 1, 3/2, 7/5, 17/12, 41/29 and 4.2×10^{-4}
   (b) 2, 9/4, 38/17, 161/72, 682/305 and 2.4×10^{-6}
   (c) 1, 2, 5/3, 7/4, 19/11 and 4.8×10^{-3}
   Largest error is $\sqrt{3}$ with 19/11.
2. (i) $[8;\ 1,\ 1,\ 2]$ (ii) $x=14+5t$ and $y=-119-43t$
3. (i) $[5;\ 1,\ 1,\ 2,\ 1,\ 2]$ (ii) $x=700+19t$ and $y=-3900-106t$
4. (a) $x=3+17t$ and $y=-15-86t$ (b) $x=-30+23t$ and $y=145-111t$
   (c) $x=-3+17t$ and $y=12-67t$
5. (a) $\dfrac{721}{228}$ (b) $\dfrac{244}{63}$ (c) $\dfrac{154}{111}$