Complete Solutions to Exercises 17(d)

1. (a) The given function is even because $f(-t)=f(t)$. For example

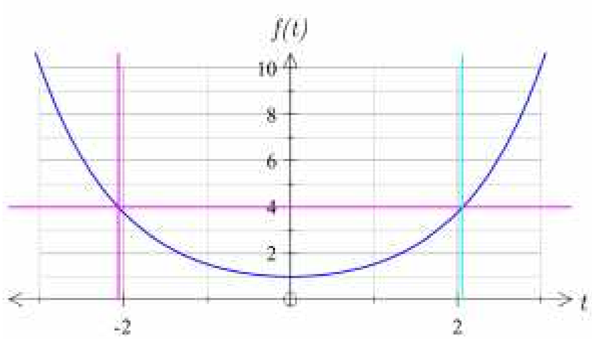
$$f(-1)=f(1)=1$$

(b) Clearly this is an odd function because

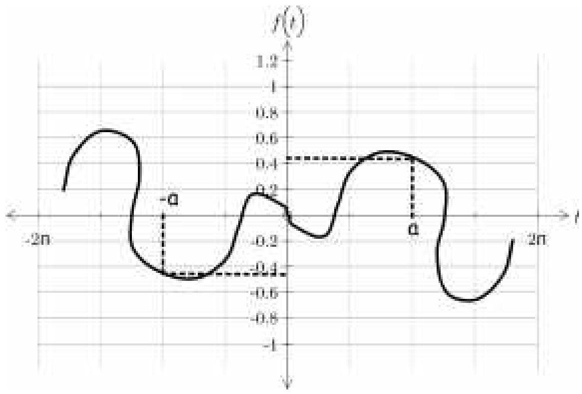
$$f\left(-\frac{\pi}{2}\right)=-2 \text{ but } f\left(\frac{\pi}{2}\right)=2 \text{ so } f\left(-\frac{\pi}{2}\right)=-f\left(\frac{\pi}{2}\right)$$

We have $f(-t)=-f(t)$.

(c) This is even because



(d) This waveform is odd because if we consider an arbitrary point $t=a$ then from the graph below we have $f(-a)=-f(a)$:



Therefore $f(-t)=-f(t)$ so the given waveform is odd.

2. (a) Clearly e^t is neither odd nor even because

$$f(-t)=e^{-t}\neq e^t,\ -e^t$$

Since $f(-t)\neq f(t)$ and $f(-t)\neq -f(t)$ so e^t cannot be odd or even.

(b) Recall $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$. Using the properties $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ we have

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$$

Let $f(t) = \tan(t)$ then by this derivation we have $f(-t) = -f(t)$ so $\tan(t)$ is an odd function.

(c) Let $f(t) = \dfrac{1}{1+t^2}$. Then

$$f(-t) = \frac{1}{1+(-t)^2} = \frac{1}{1+t^2} = f(t)$$

Because we have $f(-t) = f(t)$ so $f(t) = \dfrac{1}{1+t^2}$ is an even function.

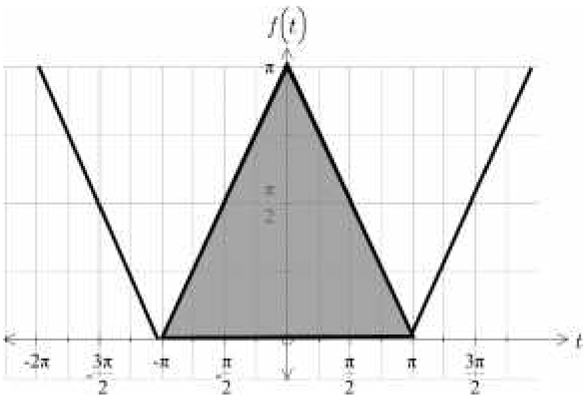
(d) Let $f(t) = e^{-t}\sin(t)$ so

$$f(-t) = e^{-(-t)}\sin(-t) = e^{t}\left[-\sin(t)\right] = -e^{t}\sin(t)$$

Since $e^{-t} \neq e^{t}$ so $f(-t) \neq f(t)$ or $f(-t) \neq -f(t)$. The given function $f(t) = e^{-t}\sin(t)$ is neither odd nor even.

3. The constant term A_0 in the Fourier series represents the average value of waveform over one complete period.

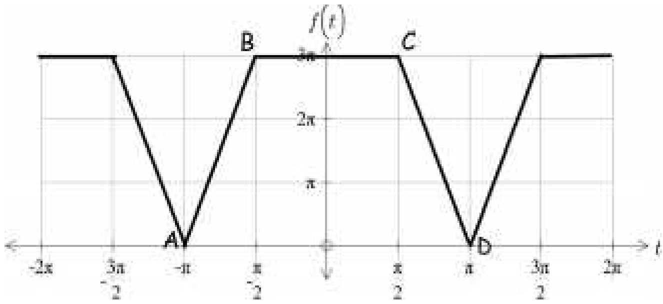
(a) In this case A_0 is the shaded area of the triangle shown divided by 2π:



$$\text{Shaded Area} = \frac{1}{2}(2\pi \times \pi) = \pi^2$$

$$A_0 = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

(b) We are given



A_0 is the area of the trapezium $ABCD$ as shown divided by 2π.

$$\text{Area of trapezium } ABCD = \frac{3\pi}{2}(2\pi + \pi) = \frac{9\pi^2}{2}$$

Therefore

$$A_0 = \frac{\text{Area}}{2\pi} = \frac{9\pi^{\not 2} / 2}{2\not\pi} = \frac{9\pi}{4}$$

4. (i) The waveform of question 3(a) is clearly an *even function* so we only need to evaluate the cosine coefficients and the constant term.

Actually we have already evaluated the constant term $A_0 = \frac{\pi}{2}$ in question 3(a).

Between 0 and π the waveform is given by $f(t) = \pi - t$. Therefore using the formula for A_k which is given by $A_k = \frac{2}{\pi}\int_0^\pi f(t)\cos(kt)\,\mathrm{d}t$ we have

$$A_k = \frac{2}{\pi}\int_0^\pi \left[(\pi - t)\cos(kt)\right]\mathrm{d}t$$

We need to use integration by parts to evaluate this integral. We have

$$u = \pi - t \qquad v' = \cos(kt)$$

$$u' = -1 \qquad v = \int \cos(kt)\,\mathrm{d}t = \frac{\sin(kt)}{k}$$

Substituting these into the integration by parts formula yields:

$$
\begin{aligned}
A_k &= \frac{2}{\pi}\int_0^{\pi}\left[(\pi - t)\cos(kt)\right]\mathrm{d}t \\
&= \frac{2}{\pi}\left\{\left[(\pi - t)\frac{\sin(kt)}{k}\right]_0^{\pi} - \int_0^{\pi}(-1)\frac{\sin(kt)}{k}\,\mathrm{d}t\right\} \\
&= \frac{2}{\pi}\left\{\underbrace{\left[\underbrace{(\pi - \pi)}_{=0}\frac{\sin(k\pi)}{k} - 0\right]}_{=0} + \frac{1}{k}\left[-\frac{\cos(kt)}{k}\right]_0^{\pi}\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{k^2}\left[\cos(k\pi) - \cos(0)\right]\right\} = -\frac{2}{k^2\pi}\left[\cos(k\pi) - 1\right]
\end{aligned}
$$

We have

$$A_k = -\frac{2}{k^2\pi}\left[\cos(k\pi) - 1\right] \qquad (*)$$

What is the value of $\cos(k\pi)$ *?*

We have the well-known trigonometric result:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

If k is even then substituting $\cos(k\pi) = 1$ into (*) gives

$$A_k = -\frac{2}{k^2\pi}\left[1 - 1\right] = 0$$

If k is odd then substituting $\cos(k\pi) = -1$ into (*) gives

$$A_k = -\frac{2}{k^2\pi}\left[-1 - 1\right] = \frac{4}{k^2\pi}$$

In the Fourier series of the given waveform $f(t)$ we only have the constant term and the odd cosine terms. We substitute these into

$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

We have

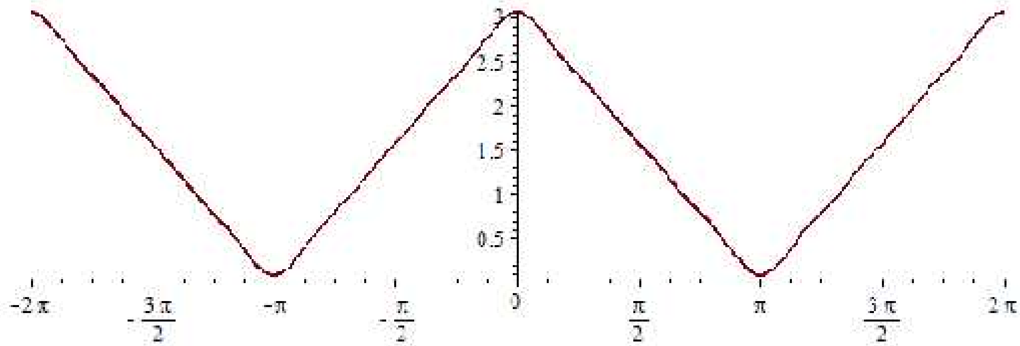
$$
\begin{aligned}
f(t) &= \frac{\pi}{2} + \frac{4}{\pi}\cos(t) + \frac{4}{9\pi}\cos(3t) + \frac{4}{25\pi}\cos(5t) + \frac{4}{49\pi}\cos(7t) + \cdots \\
&= \frac{\pi}{2} + \frac{4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \frac{\cos(7t)}{49} + \cdots\right]
\end{aligned}
$$

Here is the Maple output for this waveform:

> $f := t \to \frac{\pi}{2} + \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3\,t)}{9} + \frac{\cos(5\,t)}{25} + \frac{\cos(7\,t)}{49}\right)$

$$t \to \frac{1}{2}\pi + \frac{4\left(\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \frac{1}{49}\cos(7t)\right)}{\pi}$$

$plot(f, -2\pi..2\pi)$



(ii) We need to deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$. *How?*

By substituting $t = 0$ into our derived Fourier series of part (i). By looking at the given graph we have $f(0) = \pi$. Putting $t = 0$ into the Fourier series gives

$$f(0) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos(0) + \frac{\cos(0)}{9} + \frac{\cos(0)}{25} + \frac{\cos(0)}{49} + \cdots\right]$$

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] \qquad \left[\text{Because } \pi - \frac{\pi}{2} = \frac{\pi}{2}\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \qquad [\text{Transposing}]$$

Hence we have shown our required result.

5. (a) We need to show that $\cos(kt)$ is an even function. *What is an even function?*

$$f(-x) = f(x)$$

Using the trigonometry property $\cos(-x) = \cos(x)$ we have for $x = kt$

$$\cos(-kt) = \cos(kt)$$

Hence $\cos(kt)$ is an even function.

(b) Required to show that $\sin(kt)$ is odd. An odd function has the property

$$f(-x) = f(x)$$

Applying the trigonometry result $\sin(-x) = -\sin(x)$ we have

$$\sin(-kt) = -\sin(kt)$$

Therefore $\sin(kt)$ is an odd function.

6. a) We need to show that for an odd function with period 2π we have the following Fourier coefficients; $A_0 = A_k = 0$ and $B_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t$.

Clearly the average value over a period of 2π for an odd function is zero, so $A_0 = 0$.

Let $f(t)$ be an arbitrary odd function. By the previous question 5(a) we have $\cos(kt)$ is an even function. Using the formula for A_k;

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

The integrand is $f(t)\cos(kt)$ which is $\text{odd}\times\text{even}$ and from the main text we have $\text{odd}\times\text{even} = \text{odd}$. We have

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\text{odd function}\right]\mathrm{d}t = 0$$

To show the sine coefficients we use the formula for B_k:

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

Remember we are considering an odd function so

$$f(-t) = -f(t)$$

Evaluating B_k we have

$$\begin{aligned} B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\left\{\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0} f(t)\sin(kt)\,\mathrm{d}t\right\} \qquad (\dagger) \end{aligned}$$

Consider the second integral on the right hand side; $\int_{-\pi}^{0} f(t)\sin(kt)\,\mathrm{d}t$

Let $x = -t$ then $\frac{\mathrm{d}x}{\mathrm{d}t} = -1 \quad \Rightarrow \; \mathrm{d}x = -\mathrm{d}t$. By changing to x variable we have

$$\int_{-\pi}^{0} f(t)\sin(kt)\,\mathrm{d}t = \int_{\pi}^{0} f(-x)\sin(-kx)(-\mathrm{d}x)$$

$$= \int_{\pi}^{0} f(-x)\sin(kx)\,\mathrm{d}x \qquad \left[\text{Because } \sin(-y) = -\sin(y)\right]$$

$$= -\int_{\pi}^{0} f(x)\sin(kx)\,\mathrm{d}x \qquad \left[\begin{array}{l}\text{Because we have odd function}\\ f(-x) = -f(x)\end{array}\right]$$

$$= \int_{0}^{\pi} f(x)\sin(kx)\,\mathrm{d}x \qquad \left[\text{Applying } \int_{a}^{b} f(y)\,\mathrm{d}y = -\int_{b}^{a} f(y)\,\mathrm{d}y\right]$$

Placing this on the right hand side of (†) gives

$$B_k = \frac{1}{\pi}\left\{\int_{0}^{\pi} f(t)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0} f(t)\sin(kt)\,\mathrm{d}t\right\}$$

$$= \frac{1}{\pi}\left\{\int_{0}^{\pi} f(t)\sin(kt)\,\mathrm{d}t + \int_{0}^{\pi} f(x)\sin(kx)\,\mathrm{d}x\right\}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

This is our required result.

(b) Similarly we need to show that for an even function with period 2π we have the following Fourier coefficients; $B_k = 0$ and

$$A_k = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

First we show $B_k = 0$.

Let $f(t)$ be an arbitrary even function. By the previous question 5(b) we have $\sin(kt)$ is an odd function. Using the formula for B_k and the properties of odd and even functions we have

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(t)\sin(kt)\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[(\text{even function})\times(\text{odd function})\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\text{odd function}\right]\mathrm{d}t = 0$$

Now we show $A_k = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$.

To show the cosine coefficients we use the formula for A_k:

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

Evaluating A_k we have

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\left\{\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0} f(t)\cos(kt)\,\mathrm{d}t\right\} \qquad (\dagger\dagger) \end{aligned}$$

Consider the second integral on the right hand side; $\int_{-\pi}^{0} f(t)\cos(kt)\,\mathrm{d}t$

Let $x = -t$ then $\frac{\mathrm{d}x}{\mathrm{d}t} = -1 \Rightarrow \mathrm{d}x = -\mathrm{d}t$. By changing to x variable we have

$$\begin{aligned} \int_{-\pi}^{0} f(t)\cos(kt)\,\mathrm{d}t &= \int_{\pi}^{0} f(-x)\cos(-kx)(-\mathrm{d}x) \\ &= -\int_{\pi}^{0} f(-x)\cos(kx)\,\mathrm{d}x \qquad \left[\text{Because } \cos(-y) = \cos(y)\right] \\ &= -\int_{\pi}^{0} f(x)\cos(kx)\,\mathrm{d}x \qquad \left[\begin{array}{l}\text{Because we have even function} \\ f(-x) = f(x)\end{array}\right] \\ &= --\int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \qquad \left[\text{Applying } \int_a^b f(y)\,\mathrm{d}y = -\int_b^a f(y)\,\mathrm{d}y\right] \\ &= \int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \end{aligned}$$

Placing this $\int_{-\pi}^{0} f(t)\cos(kt)\,\mathrm{d}t = \int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x$ on the right hand side of ($\dagger\dagger$) gives

$$\begin{aligned} A_k &= \frac{1}{\pi}\left\{\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0} f(t)\cos(kt)\,\mathrm{d}t\right\} \\ &= \frac{1}{\pi}\left\{\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t + \int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x\right\} \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \end{aligned}$$

Hence we have our required result.

7. Since the given function is odd so we only need to consider the sine Fourier coefficients which are given by

(17.13)	$B_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t$

We are given that $f(t) = t^3$ between $-\pi$ and π so we need to evaluate

$$B_k = \frac{2}{\pi}\int_0^{\pi}\left[t^3 \sin(kt)\right]\mathrm{d}t \qquad (*)$$

By using the given integral $\int_0^{\pi}\left[x^3 \sin(nx)\right]\mathrm{d}x = \frac{\left(6\pi - n^2\pi^3\right)}{n^3}\cos(n\pi)$ with $x = t$ and $n = k$ we have

$$\int_0^{\pi} t^3 \sin(kt)\,\mathrm{d}t = \frac{\left(6\pi - k^2\pi^3\right)}{k^3}\cos(k\pi)$$

Substituting this into (*) yields

$$B_k = \frac{2}{\cancel{\pi}}\left[\frac{\left(6\cancel{\pi} - k^2\pi^{\cancel{3}}\right)}{k^3}\cos(k\pi)\right] = \frac{2}{k^3}\left(6 - k^2\pi^2\right)\cos(k\pi)$$

Recall that $\cos(k\pi) = 1$ if k is even and $\cos(k\pi) = -1$ if k is odd. Evaluating the first five B_k coefficients:

For $k = 1$ we have

$$B_1 = \frac{2}{1^3}\left(6 - 1^2\pi^2\right)\cos(\pi) = 2\left(6 - \pi^2\right)(-1) = -2\left(6 - \pi^2\right)$$

For $k = 2$ we have

$$B_2 = \frac{2}{2^3}\left(6 - 2^2\pi^2\right)\cos(2\pi) = \frac{2}{2^3}\left(6 - 4\pi^2\right)$$

For $k = 3$ we have

$$B_3 = \frac{2}{3^3}\left(6 - 3^2\pi^2\right)\cos(3\pi) = \frac{2}{3^3}\left(6 - 9\pi^2\right)(-1) = -\frac{2}{3^3}\left(6 - 9\pi^2\right)$$

For $k = 4$ we have

$$B_4 = \frac{2}{4^3}\left(6 - 4^2\pi^2\right)\cos(4\pi) = \frac{2}{4^3}\left(6 - 16\pi^2\right)$$

For $k = 5$ we have

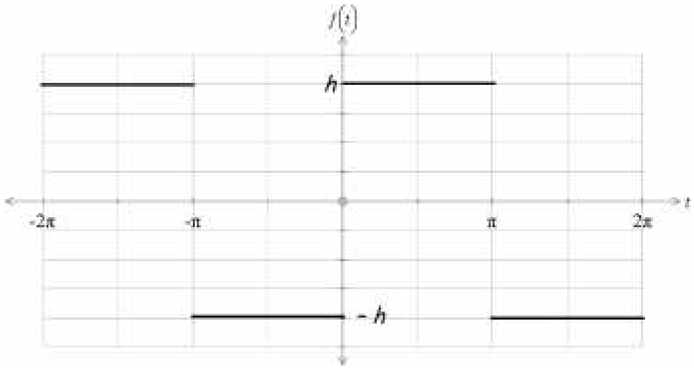
$$B_5 = \frac{2}{5^3}\left(6 - 5^2\pi^2\right)\cos(5\pi) = \frac{2}{5^3}\left(6 - 25\pi^2\right)(-1) = -\frac{2}{5^3}\left(6 - 25\pi^2\right)$$

Putting these into the Fourier series $f(t) = B_1 \sin(t) + B_2 \sin(2t) + \ldots$ gives

$$f(t) = -2(6-\pi^2)\sin(t) + \frac{2}{2^3}(6-4\pi^2)\sin(2t) - \frac{2}{3^3}(6-9\pi^2)\sin(3t) + \frac{2}{4^3}(6-16\pi^2)\sin(4t) - \frac{2}{5^3}(6-25\pi^2)\sin(5t) + \ldots$$

$$= -2\left[(6-\pi^2)\sin(t) - (6-4\pi^2)\frac{\sin(2t)}{2^3} + (6-9\pi^2)\frac{\sin(3t)}{3^3} - (6-16\pi^2)\frac{\sin(4t)}{4^3} + (6-25\pi^2)\frac{\sin(5t)}{5^3} - \ldots\right]$$

8. We are given an odd function as you observe from the graph:



Therefore we only have sine terms (B_k) in the Fourier series of $f(t)$. These are given by

(17.13) $$B_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

We have $f(t) = h$ for t between 0 and π. Substituting $f(t) = h$ into this formula:

$$\begin{aligned} B_k &= \frac{2}{\pi}\int_0^{\pi} h\sin(kt)\,\mathrm{d}t \\ &= \frac{2h}{\pi}\int_0^{\pi}\sin(kt)\,\mathrm{d}t && \left[\text{Taking out the constant } h\right] \\ &= -\frac{2h}{k\pi}\left[\cos(kt)\right]_0^{\pi} && \left[\text{Because } \int \sin(kt)\,\mathrm{d}t = -\frac{1}{k}\cos(kt)\right] \\ &= -\frac{2h}{k\pi}\left[\cos(k\pi) - 1\right] && \left[\text{Because } \cos(0) = 1\right] \end{aligned}$$

We need to use the trigonometry result:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

If k is even then substituting $\cos(k\pi) = 1$ into the above derivation

$B_k = -\frac{2h}{k\pi}\left[\cos(k\pi) - 1\right]$ gives

$$B_k = -\frac{2h}{k\pi}\left[1-1\right] = 0$$

If k is odd then substituting $\cos\left(k\pi\right) = -1$ into the above derivation

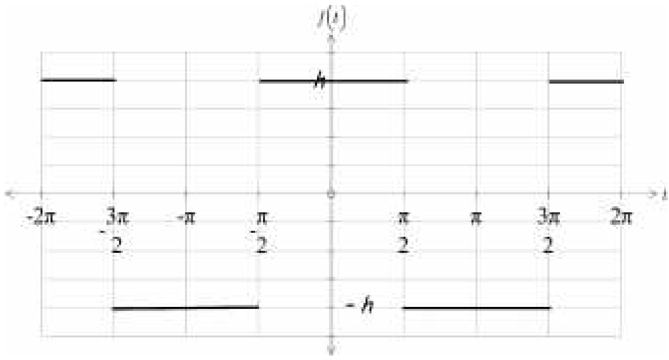
$B_k = -\frac{2h}{k\pi}\left[\cos\left(k\pi\right)-1\right]$ gives

$$B_k = -\frac{2h}{k\pi}\left[-1-1\right] = \frac{4h}{k\pi}$$

Hence the Fourier series of $f\left(t\right)$ only consists of odd sine terms and we have

$$f\left(t\right) = \frac{4h}{\pi}\left[\sin\left(t\right) + \frac{\sin\left(3t\right)}{3} + \frac{\sin\left(5t\right)}{5} + \frac{\sin\left(7t\right)}{7} + \cdots\right]$$

9. We are given the waveform:



This is an even function. Clearly the constant term is 0 because the average value of the function over a period of 2π is 0. We have $A_0 = 0$.

For the cosine coefficients we use the formula:

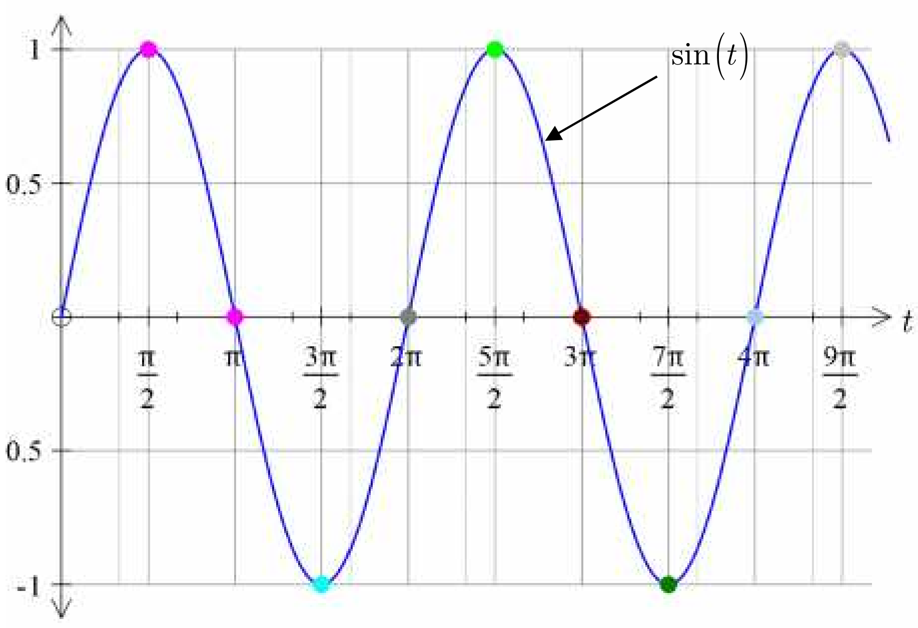
(17.15) $$A_k = \frac{2}{\pi}\int_0^{\pi} f\left(t\right)\cos\left(kt\right)\,\mathrm{d}t$$

Note that $f\left(t\right) = h$ between 0 and $\pi/2$ and $f\left(t\right) = -h$ between $\pi/2$ and π.

Substituting this into formula (17.15) we have

$$\begin{aligned}
A_k &= \frac{2}{\pi}\left[\int_0^{\pi/2} h\cos\left(kt\right)\,\mathrm{d}t + \int_{\pi/2}^{\pi}\left(-h\right)\cos\left(kt\right)\,\mathrm{d}t\right] \\
&= \frac{2h}{\pi}\left[\frac{1}{k}\left[\sin\left(kt\right)\right]_0^{\pi/2} - \frac{1}{k}\left[\sin\left(kt\right)\right]_{\pi/2}^{\pi}\right] \\
&= \frac{2h}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right) - 0\right] - \left[\underbrace{\sin\left(k\pi\right)}_{=0} - \sin\left(k\frac{\pi}{2}\right)\right] \\
&= \frac{2h}{k\pi}\left[2\sin\left(k\frac{\pi}{2}\right)\right] = \frac{4h}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right)\right]
\end{aligned}$$

From the sine graph:



We have the following result:

($) $$\sin\left(k\frac{\pi}{2}\right)=\begin{cases}1 & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -1 & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

Substituting this into $A_k=\frac{4h}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right)\right]$ gives

$$A_k=\frac{4h}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right)\right]=\begin{cases}4h/k\pi & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -4h/k\pi & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

Putting this into the general Fourier series

$$f(t)=A_0+A_1\cos(t)+A_2\cos(2t)+\ldots+B_1\sin(t)+B_2\sin(2t)+\ldots$$

Yields

$$\begin{aligned}f(t)&=0+\frac{4h}{\pi}\cos(t)+(0)\cos(2t)-\frac{4h}{3\pi}\cos(3t)+(0)\cos(4t)+\frac{4h}{5\pi}\cos(5t)+0-\frac{4h}{7\pi}\cos(7t)+\cdots\\ &=\frac{4h}{\pi}\left[\cos(t)-\frac{\cos(3t)}{3}+\frac{\cos(5t)}{5}-\frac{\cos(7t)}{7}+\cdots\right]\end{aligned}$$

By substituting $h=\pi$ into this series we have

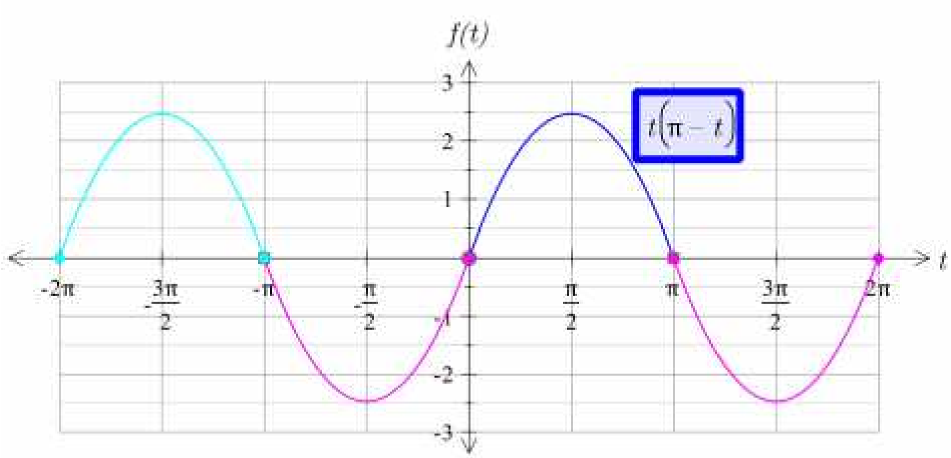
$$f(t)=\frac{4\cancel{\pi}}{\cancel{\pi}}\left[\cos(t)-\frac{\cos(3t)}{3}+\frac{\cos(5t)}{5}-\frac{\cos(7t)}{7}+\cdots\right] \qquad (*)$$

To deduce the given series we need to substitute $t = 0$ into (*). *What is* $f(0)$ *equal to?*

By the given graph we have $f(0) = h = \pi$. Substituting $t = 0$ into the right hand side of (*) yields

$$f(0) = 4\left[\cos(0) - \frac{\cos(0)}{3} + \frac{\cos(0)}{5} - \frac{\cos(0)}{7} + \frac{\cos(0)}{9} - \frac{\cos(0)}{11} + \cdots\right]$$
$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right]$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

10. (i) We are given:



This is an odd waveform so we only need to evaluate the sine coefficients which are given by

(17.13) $$B_k = \frac{2}{\pi}\int_0^\pi f(t)\sin(kt)\,\mathrm{d}t$$

Between 0 and π we have $f(t) = t(\pi - t) = \pi t - t^2$. We have

$$B_k = \frac{2}{\pi}\int_0^\pi \left[\left(\pi t - t^2\right)\sin(kt)\right]\mathrm{d}t \qquad (*)$$

We need to use integration by parts to evaluate this integral. Let

$$u = \pi t - t^2 \qquad v' = \sin(kt)$$
$$u' = \pi - 2t \qquad v = \int \sin(kt)\,\mathrm{d}t = -\frac{1}{k}\cos(kt)$$

Applying the integration by parts formula gives

$$\int_0^\pi \left[\left(\pi t - t^2\right)\sin\left(kt\right)\right] \mathrm{d}t = uv - \int u'v \, \mathrm{d}t$$

$$= -\frac{1}{k}\left[\left(\pi t - t^2\right)\cos\left(kt\right)\right]_0^\pi + \frac{1}{k}\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t$$

$$= -\frac{1}{k}\left[0\cos\left(kt\right) - 0\cos\left(kt\right)\right] + \frac{1}{k}\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t$$

$$= \frac{1}{k}\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t$$

We have

$$\int_0^\pi \left[\left(\pi t - t^2\right)\sin\left(kt\right)\right] \mathrm{d}t = \frac{1}{k}\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t \qquad (**)$$

How do we find the integral on the right hand side?

Using integration by parts again. Let

$$u = \pi - 2t \qquad v' = \cos\left(kt\right)$$

$$u' = -2 \qquad v = \int \cos\left(kt\right) \mathrm{d}t = \frac{1}{k}\sin\left(kt\right)$$

We have

$$\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t = uv - \int u'v \, \mathrm{d}t$$

$$= \frac{1}{k}\left[\left(\pi - 2t\right)\sin\left(kt\right)\right]_0^\pi - \frac{1}{k}\int_0^\pi \left[-2\sin\left(kt\right)\right]\mathrm{d}t$$

$$= \frac{1}{k}\underbrace{\left[0\right]}_{\text{because } \sin\left(k\pi\right)=0=\sin\left(0\right)} + \frac{2}{k}\left[-\frac{\cos\left(kt\right)}{k}\right]_0^\pi$$

$$= -\frac{2}{k^2}\left[\cos\left(k\pi\right) - \cos\left(0\right)\right] = -\frac{2}{k^2}\left[\cos\left(k\pi\right) - 1\right]$$

Substituting this result $\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t = -\frac{2}{k^2}\left[\cos\left(k\pi\right) - 1\right]$ into (**)

gives

$$\int_0^\pi \left[\left(\pi t - t^2\right)\sin\left(kt\right)\right] \mathrm{d}t = \frac{1}{k}\int_0^\pi \left[\left(\pi - 2t\right)\cos\left(kt\right)\right]\mathrm{d}t$$

$$= \frac{1}{k}\left(-\frac{2}{k^2}\left[\cos\left(k\pi\right) - 1\right]\right) = -\frac{2}{k^3}\left[\cos\left(k\pi\right) - 1\right]$$

Putting this $\int_0^\pi \left[\left(\pi t - t^2\right)\sin\left(kt\right)\right] \mathrm{d}t = -\frac{2}{k^3}\left[\cos\left(k\pi\right) - 1\right]$ into (*) yields

$$B_k = \frac{2}{\pi}\int_0^{\pi}\left[\left(\pi t - t^2\right)\sin\left(kt\right)\right]\mathrm{d}t$$
$$= \frac{2}{\pi}\left(-\frac{2}{k^3}\left[\cos\left(k\pi\right)-1\right]\right) = -\frac{4}{k^3\pi}\left[\cos\left(k\pi\right)-1\right]$$

Recall $\cos\left(k\pi\right) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd}\end{cases}$. If k is even we have

$$B_k = -\frac{4}{k^3\pi}\left[\cos\left(k\pi\right)-1\right] = -\frac{4}{k^3\pi}\left[1-1\right] = 0$$

If k is odd we have

$$B_k = -\frac{4}{k^3\pi}\left[\cos\left(k\pi\right)-1\right] = -\frac{4}{k^3\pi}\left[-1-1\right] = \frac{8}{k^3\pi}$$

Hence the Fourier series only contains odd sine terms.

The general Fourier series is

$$f\left(t\right) = A_0 + A_1\cos\left(t\right) + A_2\cos\left(2t\right) + \ldots + B_1\sin\left(t\right) + B_2\sin\left(2t\right) + \ldots$$

Substituting $A_0 = A_k = 0$ and $B_k = \begin{cases} 0 & \text{if } k = \text{even} \\ \dfrac{8}{k^3\pi} & \text{if } k = \text{odd}\end{cases}$ into this gives

$$f\left(t\right) = \frac{8}{\pi}\sin\left(t\right) + \frac{8}{3^3\pi}\sin\left(3t\right) + \frac{8}{5^3\pi}\sin\left(5t\right) + \frac{8}{7^3\pi}\sin\left(7t\right) + \ldots$$
$$= \frac{8}{\pi}\left[\sin\left(t\right) + \frac{\sin\left(3t\right)}{3^3} + \frac{\sin\left(5t\right)}{5^3} + \frac{\sin\left(7t\right)}{7^3} + \ldots\right]$$

(ii) Substituting $t = \dfrac{\pi}{2}$ into the Fourier series derived in part (i) gives

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{\sin\left(3\pi/2\right)}{3^3} + \frac{\sin\left(5\pi/2\right)}{5^3} + \frac{\sin\left(7\pi/2\right)}{7^3} + \ldots\right] \qquad (\dagger)$$

Our given function is $f\left(t\right) = t\left(\pi - t\right)$ when $0 < t < \pi$. Since $0 < \dfrac{\pi}{2} < \pi$ so we have

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{\pi^2}{4}$$

Putting this into the left hand side of (†) and evaluating the right hand side we have

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{\sin\left(3\pi / 2\right)}{3^3} + \frac{\sin\left(5\pi / 2\right)}{5^3} + \frac{\sin\left(7\pi / 2\right)}{7^3} + \ldots\right]$$

$$= \frac{8}{\pi}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\right]$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

Hence we have our required result $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$.