SECTION B **Injective and Surjective functions**

By the end of this section you will be able to

- understand and show that a given function is an injective function
- understand and show that a given function is a surjective function

B1 Injective Functions

Example 9

Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2$. Determine the images of

$$-3,\ -2,\ -1,\ 0,\ 1,\ 2 \text{ and } 3$$

Solution

Substituting these values into the given function $f(x)=x^2$ yields

Plotting these on a diagram:

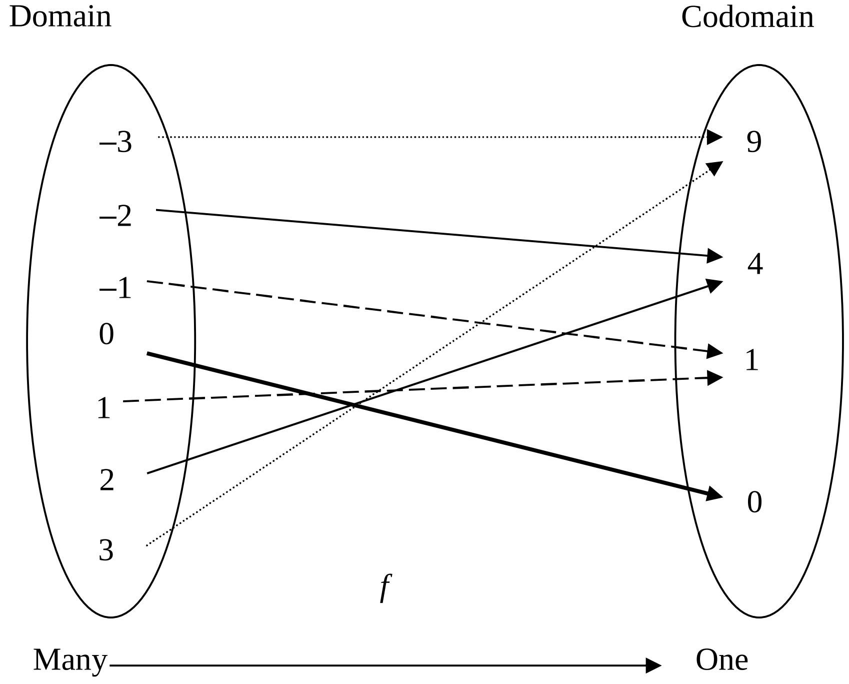


Fig 12

This function, $f(x)=x^2$, is an example of a **many to one** function. Many from the domain arrive at a single destination.

Clearly the given f is a function because every number in the domain has been assigned by f to a **unique** number in the codomain.

Example 10

Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=2x$. Determine the images of

$$-3,\quad -2,\quad -1,\quad 0,\quad 1,\quad 2 \text{ and } 3$$

Solution

Substituting these values into the given function $f(x)=2x$ yields

$$\begin{aligned} f(-3) &= 2\times(-3) = -6 \\ f(-2) &= 2\times(-2) = -4 \\ f(-1) &= 2\times(-1) = -2 \\ f(0) &= 2\times 0 = 0 \\ f(1) &= 2\times 1 = 2 \\ f(2) &= 2\times 2 = 4 \\ f(3) &= 2\times 3 = 6 \end{aligned}$$

Putting this on a diagram:

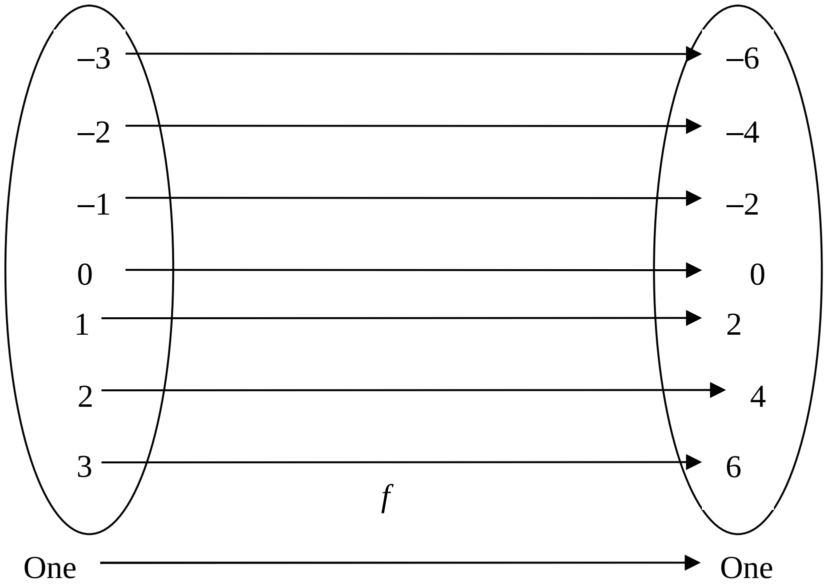


Fig 13

This function is an example of a **one to one** function.

Functions can be one to one or many to one but **not** one to many or many to many. *Why not?*

Because a function must arrive at only one station.

One to one functions are particularly important and we examine them in the rest of this subsection.

One to one functions are also called **injective** functions. We say a function $f:A\to B$ is an **injection** if it is a one to one function. The formal definition is:

Definition (3.2)

A function $f: A \to B$ is a **one to one** or **injective** function if and only if for all $x, \quad y$ in A

$$f(x) = f(y) \quad \Rightarrow \quad x = y$$

In the next few examples we use this definition to test whether given functions are one to one (or injective).

Example 11

Let $A = \{x \in \mathbb{R} \mid x \neq 3\}$ and a function $f: A \to \mathbb{R}$ be defined by $f(x) = \dfrac{2x}{x-3}$.

Decide and show whether the function f is an injective function.

Solution

What does the term injective mean?

One to one. If we show that the given function is one to one then we can conclude that it is an injective function. *How do we show f is one to one?* By using definition (3.2). We assume $f(x) = f(y)$ and need to show that $x = y$. *What is $f(x)$ and $f(y)$ equal to in this case?*

$$f(x) = \frac{2x}{x-3} \text{ and } f(y) = \frac{2y}{y-3}$$

Therefore $f(x) = \dfrac{2x}{x-3} = \dfrac{2y}{y-3} = f(y)$. We have

$$\frac{2x}{x-3} = \frac{2y}{y-3}$$

$$2x(y-3) = 2y(x-3) \qquad [\text{Multiplying by } (y-3)(x-3)]$$

$$2xy - 6x = 2xy - 6y \qquad [\text{Expanding Brackets}]$$

$$-6x = -6y \qquad [\text{Simplifying}]$$

$$x = y \qquad [\text{Dividing through by } -6]$$

Since we have $x = y$ therefore we conclude that the given function, f, is a one to one function or an injective function.

Example 12

Let $f: \mathbb{R}^+ \to \mathbb{R}$ be given by $f(x) = x^2 + 2$. Show that the function f is an injection. (A function which is injective is called an **injection**).

Solution

What does the term injection mean?

One to one. So we need to show that the given function is one to one. *How?*
By using definition (3.2). We assume $f(x) = f(y)$ and show that this does lead to $x = y$.
What is $f(x)$ and $f(y)$ equal to in this case?

$$f(x) = x^2 + 2 \quad \text{and} \quad f(y) = y^2 + 2$$

Similar to the above example we have

$$x^2 + 2 = y^2 + 2$$
$$x^2 = y^2 \quad \Rightarrow \quad x = \pm y$$

Remember $x = \pm y$ means $x = y$ or $x = -y$.
But for f to be one to one function we need $x = y$ and **not** $x = -y$. *What mistake have we made ?*
So far we have made no mistake. Examine the domain, $\mathbb{R}^+$, which means that x and y can only have **positive** real values. *Why?*
Because both x and y are members of the domain $\mathbb{R}^+$.
Hence x cannot equal $-y$ that is $x \neq -y$. Therefore $x = y$.
This means we have shown that $f(x) = f(y) \Rightarrow x = y$
Hence by definition (3.2) the given function f is an injection on the stated domain and codomain.

Example 13

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Prove that the function f is **not** injective.

<u>Solution</u>

*What does the term **not injective** mean?*
We need to show that the given function, $f(x) = x^2$, is **not** one to one.
*How do we show that $f(x) = x^2$ is **not** one to one?*
By using definition (3.2). We assume $f(x) = f(y)$ and show that this does **not** lead to $x = y$. *What does $f(x) = f(y)$ mean in this case?*
It means that $x^2 = y^2$ because $f(x) = x^2$ and $f(y) = y^2$. Hence we have

$$x^2 = y^2$$
$$\sqrt{x^2} = \sqrt{y^2} \quad \Rightarrow \quad x = \pm y$$

Remember $x = \pm y$ means $x = y$ or $x = -y$. This time we **cannot** dismiss the answer $x = -y$ because the domain, $\mathbb{R}$, is the set of all real numbers and therefore both, $x = -y$ and $x = y$, are in the domain.

Since we do **not** have $x = y$ as the only Solution therefore we conclude that the given function, f, is not one to one or it is **not** an injective function.

Sometimes it is easier to deal with the contrapositive of definition (3.2). Remember the contrapositive of a proposition is equivalent to the proposition. *What is the contrapositive of*

$$\textit{for all } x,\ \ y \textit{ in } A \qquad f(x) = f(y) \ \Rightarrow\ x = y\,?$$

Contrapositive is

$$\text{for all } x,\ \ y \text{ in } A \qquad x \neq y \ \Rightarrow\ f(x) \neq f(y)$$

This means that for all x and y in A (the domain) such that if x does **not** equal y then $f(x)$ does **not** equal $f(y)$. You might find it easier to use this to prove that a given function is injective or *not* injective.

For example to prove that $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x) = 2x+1$ is one to one (injective) we assume $x \neq y$. This implies

$$\begin{aligned} & 2x \neq 2y \\ \Rightarrow\ & 2x+1 \neq 2y+1 \\ \Rightarrow\ & f(x) \neq f(y) \qquad \left[\text{Because } f(x) = 2x+1 \text{ and } f(y) = 2y+1\right] \end{aligned}$$

Since $x \neq y \ \Rightarrow\ f(x) \neq f(y)$ we conclude that $f(x) = 2x+1$ is a one to one (or injective) function.

Consider another example, prove that $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = x^2$ is **not** injective. *How do we prove this using the contrapositive of (3.2)?*

We only need to prove that for some $x,\ y$ in the domain, $\mathbb{R}$, such that

$$x \neq y \ \Rightarrow\ f(x) = f(y)$$

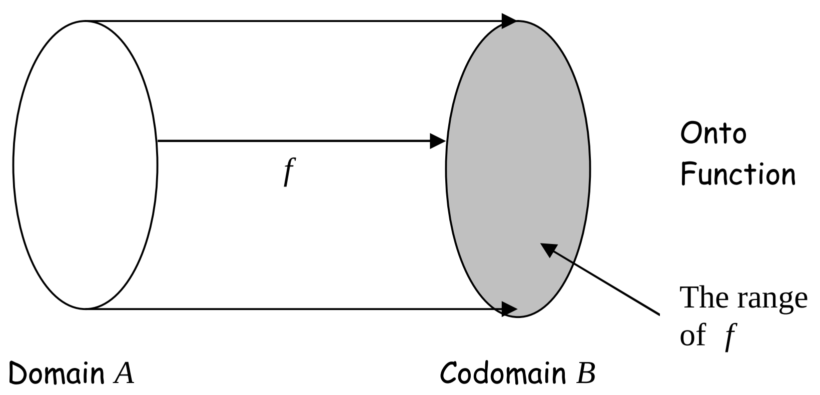
For **not** injection we show that there are some x and y in the domain such that x does **not equal** y but $f(x) = f(y)$.

Let $x = -3$ and $y = 3$ then $-3 \neq 3$ but $f(-3) = 9 = f(3)$. We have shown that there are x and y in the domain, $\mathbb{R}$, such that $x \neq y$ but $f(x) = f(y)$. We conclude that $f(x) = x^2$ is **not** an injection on the stated domain and codomain, $\mathbb{R}$. This is a many to one function and **not** one to one as discussed in Example 9.

B2 Surjective Functions

Let $f: A \to B$ be a function. The range of f is a subset of the codomain B. If the range is equal to the codomain then we say the function f is **onto**.

Examining the concept of onto functions by diagrams we have:



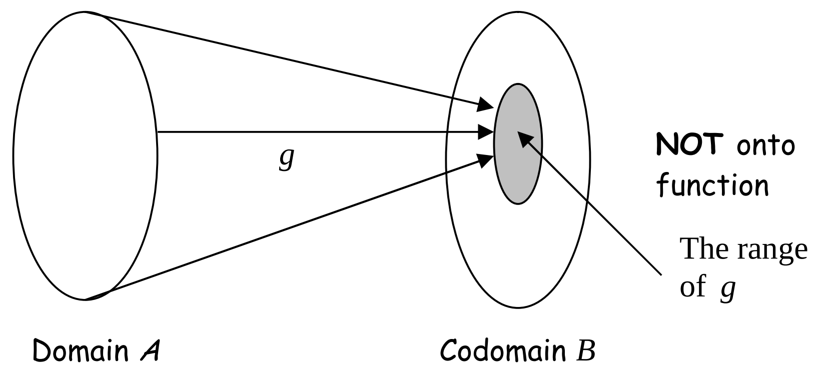


Fig 14

Fig 14 shows two functions f and g where f is an onto function and g is **not** onto.

An **onto** function is also called a **surjective** function or just **surjection**.

A formal definition is:

Definition (3.3)

A function $f: A \to B$ is an **onto** or **surjective** function if and only if the range of f equals B.

This means that for every element in the codomain, B, there is an element in the domain, A, which is transformed to it via f.

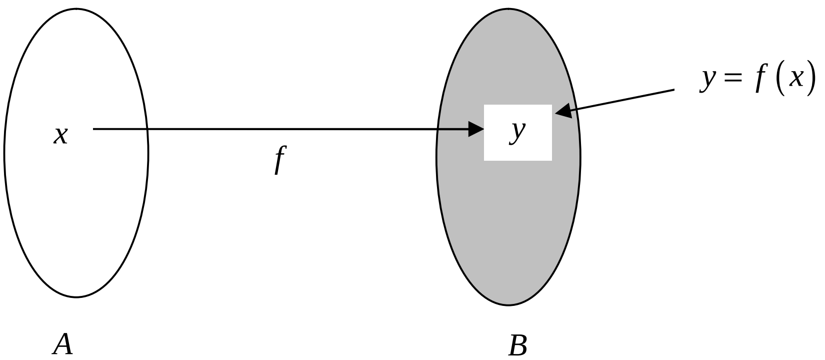


Fig 15

To prove that a given function is onto (surjective) we need to show that for every y in B there is an x in A such that $f(x) = y$.

The procedure for proving that a function given by a formula is surjective is:

1) Let y be in the codomain such that $y = f(x)$.
2) Solve the equation $y = f(x)$ for x.
3) Check that x found in part 2) is a member of the domain.

If you discover that the x you have found in part 2) is **not** in the domain of the function then f is **not** surjective.

Showing surjection is a lot more difficult than showing injection of a given function *f*.

Example 14

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x + 1$. Test whether *f* is surjective.

<u>Solution</u>

Let y be in the codomain then we have $y = f(x) = x + 1$. We need to solve this equation, $y = x + 1$, for x. *What is x equal to?*

$$x = y - 1$$

The element $y \in \mathbb{R}$. *How do we know this?*

Because y is a member of the codomain and the codomain is equal to $\mathbb{R}$.

Since y is real number then so is $y - 1$ which means x is a real number because $x = y - 1$. Therefore x is in the domain. *Why?*

Because the domain is the set of all real numbers, $\mathbb{R}$.

Hence we have found an x in the domain such that $y = f(x)$ so the given function f is a surjective (or onto) function.

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Example 15

Let $A = \left\{x \in \mathbb{R} \mid 0 \le x \le 5\right\}$ and $B = \left\{x \in \mathbb{R} \mid 1 \le x \le 6\right\}$ and $f : A \to B$ be a function defined by $f(x) = x + 1$. Show that f is a surjective function.

<u>Solution</u>

Let y be in the codomain. We need to find an x in A such that $y = f(x) = x + 1$ where y is in B. *How do we know y is in B?*

Because B is the codomain. *What is y?*

Since $y \in B = \left\{x \in \mathbb{R} \,\middle|\, 1 \leq x \leq 6\right\}$ therefore y is a real number between 1 and 6 (inclusive). *Can we find an* x *in* A *for such a* y *?*

$$y = x + 1$$
$$x = y - 1$$

Since $1 \leq y \leq 6$ therefore $x = y - 1$ yields $0 \leq x \leq 5$. We are given $A = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\right\}$. *Is* x *in the domain* A *?*

Yes because $A = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\right\}$ which means A is the set of all real numbers between 0 to 5 (inclusive).

Therefore we have an x in A such that $x = y - 1$ which means that we have located an x in the domain with the property $y = f(x) = x + 1$.

We conclude that the given function is a surjection on the stated domain and codomain.

Example 16

Let $A = \left\{x \in \mathbb{R} \,\middle|\, 0 \leq x \leq 5\right\}$ and $B = \left\{x \in \mathbb{R} \,\middle|\, 1 \leq x \leq 7\right\}$ and $f : A \to B$ be a function defined by $f(x) = x + 1$. Show that f is **not** a surjection.

<u>Solution</u>

For a function to be surjective we need to show that for every y in B there is an x in A such that $f(x) = y$. *How do we show that the given function is* ***not*** *surjective?*

Find a y in B such that there is **no** x in A which is assigned to y by the given function f. *Is there such a* y *in* B *?*

Yes, consider $y = 7$ clearly y is in $B = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 1 \leq x \leq 7\right\}$ and no x in $A = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\right\}$ is assigned to 7. *Why not?*

Because $x = y - 1$ and therefore $x = 7 - 1 = 6$ but 6 is **not** a member of $A = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\right\}$. *Why not?*

Because $A = \left\{x \,\middle|\, x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\right\}$ which means the set A contains all the real numbers between 0 to 5 (inclusive).

Hence we have shown that the given function, $f(x) = x + 1$, is **not** a surjection on the stated domain, A, and codomain, B.

Example 17

Let $A = \{x \in \mathbb{R} \mid x \neq 2\}$ and $B = \{x \in \mathbb{R} \mid x \neq 3\}$ and $f : A \to B$ be a function defined by $f(x) = \dfrac{2x}{x-3}$. Show that f is an onto function.

Solution

Remember an onto function, surjective function and surjection are all the same thing. To prove that the given function is onto we show that there is an x in A such that $f(x) = y$

$$y = f(x) = \frac{2x}{x-3}$$

Need to solve this equation, $y = \dfrac{2x}{x-3}$, for x:

$$y(x-3) = 2x \qquad \text{[Multiplying both sides by } (x-3)]$$
$$yx - 3y = 2x \qquad \text{[Expanding Brackets]}$$
$$yx - 2x = 3y \qquad \text{[Collecting Like Terms]}$$
$$x(y-2) = 3y \qquad \text{[Factorizing]}$$
$$x = \frac{3y}{y-2} \qquad \text{[Dividing through by } (y-2)]$$

$x = \dfrac{3y}{y-2}$ is a real number provided $y \neq 2$. Hence we have located an x in $A = \{x \in \mathbb{R} \mid x \neq 3\}$ such that $f(x) = y$. We conclude that f is an onto (or surjective) function.

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SUMMARY

A function $f : A \to B$ is a **one to one** or an **injective** function if and only if

$$f(x) = f(y) \quad \Rightarrow \quad x = y$$

The function $f : A \to B$ is **onto** or surjective if and only if the range of f is equal to the codomain, B. This means that for every $y \in B$ there is an $x \in A$ such that $f(x) = y$.