Exercise **I(d)**

1. Show that for all-natural numbers, n:

$$2+4+6+...+2n=\sum_{m=1}^{n}2m=n\left(n+1\right).$$

2. Prove that for all-natural numbers, n:

$$2+5+8+...+\left(3n-1\right)=\sum_{m=1}^{n}\left(3m-1\right)=\frac{1}{2}n\left(3n+1\right).$$

3. Prove that for all-natural numbers, n:

$$\sum_{m=1}^{n}m^3=\frac{1}{4}n^2\left(n+1\right)^2 .$$

4. Prove that for all natural numbers, n:

$$\sum_{m=1}^{n}m^3=\left(1+2+3+4+...+n\right)^2.$$

[Hint: Use the result of Question 3].

5. Prove that for all natural numbers, n:

$$\left(1\times2\right)+\left(2\times3\right)+...+n\left(n+1\right)=\sum_{m=1}^{n}m\left(m+1\right)=\frac{1}{3}n\left(n+1\right)\left(n+2\right).$$

6. Prove that for all natural numbers, n:

$$\left(1\times2\times3\right)+\left(2\times3\times4\right)+\left(3\times4\times5\right)+...+n\left(n+1\right)\left(n+2\right)=\frac{1}{4}n\left(n+1\right)\left(n+2\right)\left(n+3\right)$$

7. Show that for every natural number, n:

$$1+r+r^2+...+r^n=\sum_{m=0}^{n}r^m=\frac{1-r^{n+1}}{1-r}\qquad\left(r\neq1\right)$$

[This is the geometric series with the first term equal to 1]

8. Show that for all natural numbers, n:

$$\sum_{m=0}^{n}2^m=2^{n+1}-1.$$

9. Prove that for all natural numbers, n:

$$\sum_{m=1}^{n}\left(2m-1\right)^{3}=n^{2}\left(2n^{2}-1\right).$$

10. Prove that for all natural numbers, n:

$$\sum_{m=1}^{n}m^{4}=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^{2}+3n-1\right)}{30}.$$

11. Prove that for all natural numbers, n:

$$\sum_{m=1}^{n}m^{5}=\frac{n^{2}\left(n+1\right)^{2}\left(2n^{2}+2n-1\right)}{12}.$$

12. Prove that for all natural numbers n, 9 divides $10^{n}-1$.

13. Prove that for all natural numbers n:

$$3\,\Big|\,\left(n^{3}-n\right).$$

14. Show that for every natural number n:

$$3\,\Big|\,n\left(n+1\right)\left(n+2\right).$$

15. Prove that for all natural numbers n:

$n^{2}-n$ is an even number.

16. Prove that for all natural numbers n:

$$\sum_{m=1}^{n}ar^{m-1}=a+ar+ar^{2}+\ldots+ar^{n-1}=\frac{a\left(1-r^{n}\right)}{1-r}\qquad\left[r\neq 1\right]$$

where a and r are real numbers.

[This is a geometric series with first term equal to a and common ratio r.]

17. *Prove that for all natural numbers n we have the following trigonometric identity:

$$\sin(x) + \sin(2x) + ... + \sin(nx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

where x is a real number such that $\sin\left(\frac{x}{2}\right) \neq 0$.

18. **Prove the binomial theorem for the natural number n:

If a and b are real numbers, then the binomial theorem says that for all natural numbers n:

$$\left(a+b\right)^n = a^n + na^{n-1}b + \frac{n\left(n-1\right)}{2!}a^{n-2}b^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}a^{n-3}b^3 + ... + b^n.$$