## Complete Solutions to Exercises 17(b)

1. (a) We have $f\left(t\right)=\begin{cases}1 & 0<t<\pi \\ 0 & -\pi<t<0\end{cases}.$

The constant term $A_0$ is given by

$$
\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(t\right)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}\left(1\right)\mathrm{d}t+\int_{-\pi}^{0}\left(0\right)\mathrm{d}t\right\} && \left[\text{Splitting } f\left(t\right)\right] \\
&= \frac{1}{2\pi}\left[t\right]_0^{\pi} && \left[\text{Integrating}\right] \\
&= \frac{1}{2\pi}\left[\pi-0\right] && \left[\text{Substituting for } t\right] \\
&= \frac{1}{2\not{\pi}}\not{\pi} && \left[\text{Simplifying}\right] \\
A_0 &= \frac{1}{2}
\end{aligned}
$$

The cosine coefficients are given by

$$
\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(t\right)\cos\left(kt\right)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^{\pi}\left(1\right)\cos\left(kt\right)\mathrm{d}t && \left[\text{Replacing } f\left(t\right)\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{\sin\left(kt\right)}{k}\right]_0^{\pi}\right\} && \left[\text{Integrating by } \int\cos\left(kt\right)dt=\frac{\sin\left(kt\right)}{k}\right] \\
&= \frac{1}{k\pi}\left[\sin\left(kt\right)\right]_0^{\pi} && \left[\text{Taking out a factor of } \frac{1}{k}\right] \\
&= \frac{1}{k\pi}\left[\sin\left(k\pi\right)-\sin\left(0\right)\right] && \left[\text{Substituting the limits}\right] \\
&= \frac{1}{k\pi}\left\{\left[0-0\right]\right\}=0 && \left[\text{Because } \sin\left(k\pi\right)=0\right]
\end{aligned}
$$

We have $A_k=0$.

We also need to find the sine coefficients:

$$\begin{aligned}
B_k &= \frac{1}{\pi}\int_0^{\pi}(1)\sin(kt)\,\mathrm{d}t && \big[\text{Replacing } f(t)\big]\\
&= \frac{1}{\pi}\int_0^{\pi}\sin(kt)\,\mathrm{d}t\\
&= \frac{1}{\pi}\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} && \left[\text{Integrating by } \int\sin(kt)\,dt = -\frac{\cos(kt)}{k}\right]\\
&= -\frac{1}{k\pi}\big[\cos(kt)\big]_0^{\pi} && \left[\text{Taking out a factor of } -\frac{1}{k}\right]\\
&= -\frac{1}{k\pi}\big[\cos(k\pi)-\cos(0)\big] && \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right]\\
&= -\frac{1}{k\pi}\big[\cos(k\pi)-1\big] && \big[\text{Remember } \cos(0)=1\big]
\end{aligned}$$

If $k$ is even then $\cos(k\pi)=1$, substituting this into the last line above gives:

$$B_k = -\frac{1}{k\pi}\big[\cos(k\pi)-1\big] = -\frac{1}{k\pi}\big[1-1\big] = 0$$

If $k$ is odd then $\cos(k\pi)=-1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{1}{k\pi}\big[\cos(k\pi)-1\big] = -\frac{1}{k\pi}\big[-1-1\big] = -\frac{1}{k\pi}\big[-2\big] = \frac{2}{k\pi}$$

Putting all these coefficients together we have

$$\begin{aligned}
A_0 &= 1/2\\
A_k &= 0\\
B_k &= \begin{cases} 0 & \text{if } k \text{ is even}\\ \dfrac{2}{k\pi} & \text{if } k \text{ is odd}\end{cases}
\end{aligned}$$

Substituting these into

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

Gives

$$\begin{aligned}
f(t) &= \frac{1}{2} + \underbrace{0+0+\ldots}_{\text{No cosine terms}} + \frac{2}{\pi}\sin(t) + 0 + \frac{2}{3\pi}\sin(3t) + 0 + \frac{2}{5\pi}\sin(5t) + \ldots\\
&= \frac{1}{2} + \frac{2}{\pi}\sin(t) + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \ldots && \big[\text{Simplifying}\big]\\
&= \frac{1}{2} + \frac{2}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right] && \left[\begin{array}{l}\text{Taking out a common}\\ \text{factor of } 2/\pi\end{array}\right]
\end{aligned}$$

(ii) Substituting $t = \pi/2$ into this series gives

$$f\left(\frac{\pi}{2}\right)=\frac{1}{2}+\frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right)+\frac{\sin\left(3\pi/2\right)}{3}+\frac{\sin\left(5\pi/2\right)}{5}+\frac{\sin\left(7\pi/2\right)}{7}+\ldots\right]$$

$$1=\frac{1}{2}+\frac{2}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

$$\frac{1}{2}=\frac{2}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

(b) By examining the graph of part (b) we can split $f(t)$ into $f(t)=4$ for $t$ between 0 and $\pi$ and $f(t)=2$ for $t$ between $\pi$ and $2\pi$. We have

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,\mathrm{d}t$$

$$=\frac{1}{2\pi}\left\{\int_0^{\pi}(4)\,\mathrm{d}t+\int_{\pi}^{2\pi}(2)\,\mathrm{d}t\right\} \qquad \left[\text{Splitting } f(t)\right]$$

$$=\frac{1}{2\pi}\left\{4\left[t\right]_0^{\pi}+2\left[t\right]_{\pi}^{2\pi}\right\} \qquad \left[\text{Integrating}\right]$$

$$=\frac{1}{2\pi}\left\{4\left[\pi-0\right]+2\left[2\pi-\pi\right]\right\} \qquad \left[\text{Substituting for } t\right]$$

$$=\frac{1}{2\not\pi}\left[6\not\pi\right] \qquad \left[\text{Simplifying}\right]$$

$$A_0=3$$

For $A_k$ we use the following formula:

(17.4) $$A_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(kt)\,dt$$

Again we split $f(t)$ into 4 and 2.

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(t\right)\cos\left(kt\right)\,\mathrm{d}t$$
$$= \frac{1}{\pi}\left\{\int_0^{\pi}\left(4\right)\cos\left(kt\right)\mathrm{d}t + \int_{\pi}^{2\pi}\left(2\right)\cos\left(kt\right)\mathrm{d}t\right\} \qquad \left[\text{Replacing } f\left(t\right)\right]$$
$$= \frac{1}{\pi}\left\{4\int_0^{\pi}\cos\left(kt\right)\mathrm{d}t + 2\int_{\pi}^{2\pi}\cos\left(kt\right)\mathrm{d}t\right\}$$
$$= \frac{1}{\pi}\left\{4\left[\frac{\sin\left(kt\right)}{k}\right]_0^{\pi} + 2\left[\frac{\sin\left(kt\right)}{k}\right]_{\pi}^{2\pi}\right\} \qquad \left[\text{Integrating by } \int\cos\left(kt\right)dt = \frac{\sin\left(kt\right)}{k}\right]$$
$$= \frac{2}{k\pi}\left\{2\left[\sin\left(kt\right)\right]_0^{\pi} + \left[\sin\left(kt\right)\right]_{\pi}^{2\pi}\right\} \qquad \left[\text{Taking out a factor of } \frac{2}{k}\right]$$
$$= \frac{2}{k\pi}\left\{2\left[\sin\left(k\pi\right) - \sin\left(0\right)\right] + \left[\sin\left(2\pi k\right) - \sin\left(k\pi\right)\right]\right\} \qquad \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right]$$
$$= \frac{2}{k\pi}\left\{2\left[0-0\right] + \left[0-0\right]\right\} \qquad \left[\text{Because } \sin\left(k\pi\right) = 0\right]$$
$$A_k = 0$$

*How do we find $B_k$ ?*

Similarly by applying the formula

$$(17.5) \qquad B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(t\right)\sin\left(kt\right)\,dt$$

Splitting $f\left(t\right)$ gives

$$B_k = \frac{1}{\pi}\left\{\int_0^{\pi}\left(4\right)\sin\left(kt\right)\mathrm{d}t + \int_{\pi}^{2\pi}\left(2\right)\sin\left(kt\right)\mathrm{d}t\right\} \qquad \left[\text{Replacing } f\left(t\right)\right]$$
$$= \frac{2}{\pi}\left\{2\int_0^{\pi}\sin\left(kt\right)\mathrm{d}t + \int_{\pi}^{2\pi}\sin\left(kt\right)\mathrm{d}t\right\}$$
$$= \frac{2}{\pi}\left\{2\left[-\frac{\cos\left(kt\right)}{k}\right]_0^{\pi} + \left[-\frac{\cos\left(kt\right)}{k}\right]_{\pi}^{2\pi}\right\} \qquad \left[\text{Integrating by } \int\sin\left(kt\right)dt = -\frac{\cos\left(kt\right)}{k}\right]$$
$$= -\frac{2}{k\pi}\left\{2\left[\cos\left(kt\right)\right]_0^{\pi} + \left[\cos\left(kt\right)\right]_{\pi}^{2\pi}\right\} \qquad \left[\text{Taking out a factor of } -\frac{1}{k}\right]$$
$$= -\frac{2}{k\pi}\left\{2\left[\cos\left(k\pi\right) - \cos\left(0\right)\right] + \left[\cos\left(2\pi k\right) - \cos\left(k\pi\right)\right]\right\} \qquad \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right]$$
$$= -\frac{2}{k\pi}\left\{2\cos\left(k\pi\right) - 2 + 1 - \cos\left(k\pi\right)\right\} \qquad \left[\text{Remember } \cos\left(0\right) = \cos\left(2\pi k\right) = 1\right]$$
$$= -\frac{2}{k\pi}\left\{\cos\left(k\pi\right) - 1\right\}$$

If $k$ is even then $\cos\left(k\pi\right) = 1$, substituting this into the last line above gives:

$$B_k = -\frac{2}{k\pi}\left\{\underbrace{\cos\left(k\pi\right)}_{=1} - 1\right\} = 0 \text{ for even } k$$

If $k$ is odd then $\cos\left(k\pi\right)=-1$, again substituting this into the last line in the above derivation gives

$$B_k=-\frac{2}{k\pi}\left\{\underbrace{\cos\left(k\pi\right)}_{=-1}-1\right\}=\frac{4}{k\pi}$$

Putting all these coefficients together we have

$$\begin{aligned}A_0&=3\\A_k&=0\\B_k&=\begin{cases}0 & \text{if } k \text{ is even}\\ \dfrac{4}{k\pi} & \text{if } k \text{ is odd}\end{cases}\end{aligned}$$

Substituting these into

(17.2) $f\left(t\right)=A_0+A_1\cos\left(t\right)+A_2\cos\left(2t\right)+\cdots+B_1\sin\left(t\right)+B_2\sin\left(2t\right)+\cdots$

Gives

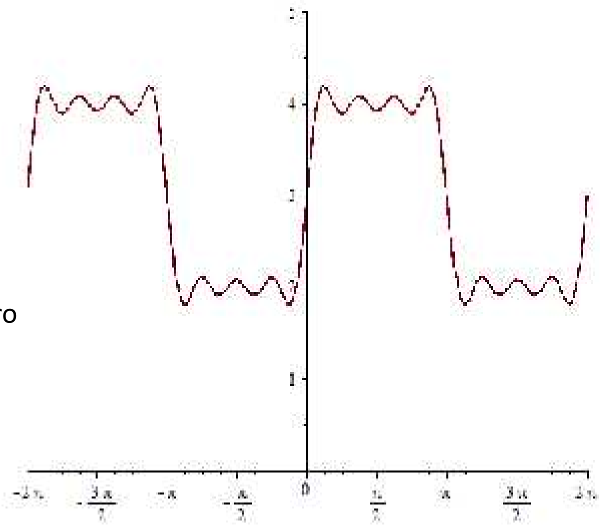
$$\begin{aligned}f\left(t\right)&=3+0+0+\ldots+\frac{4}{\pi}\sin\left(t\right)+0+\frac{4}{3\pi}\sin\left(3t\right)+0+\frac{4}{5\pi}\sin\left(5t\right)+\ldots\\&=3+\frac{4}{\pi}\left[\sin\left(t\right)+\frac{\sin\left(3t\right)}{3}+\frac{\sin\left(5t\right)}{5}+\cdots\right]\end{aligned}$$

The Maple output is as follows:

> $f:=t\rightarrow 3+\frac{4}{\pi}\left(\sin(t)+\frac{\sin(3t)}{3}+\left(\frac{\sin(5t)}{5}\right)+\frac{\sin(7t)}{7}\right)$

$$f:=t\rightarrow 3+\frac{4\left(\sin(t)+\frac{1}{3}\sin(3t)+\frac{1}{5}\sin(5t)+\frac{1}{7}\sin(7t)\right)}{\pi}$$

> $plot(f,-2\pi..2\pi,0..5)$




The graph of the Fourier series of the first 5 non-zero terms of $f\left(t\right)$.

(ii) Substituting $t=\pi/2$ into the above Fourier series:

$$f\left(\frac{\pi}{2}\right) = 3 + \frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{\sin(3\pi/2)}{3} + \frac{\sin(5\pi/2)}{5} + \frac{\sin(7\pi/2)}{7} + \cdots\right]$$

$$4 = 3 + \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

2. We can split $f(t)$ into $f(t) = 2$ for $t$ between 0 and $\pi$ and $f(t) = -2$ for $t$ between $-\pi$ and 0. We have

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}(2)\,\mathrm{d}t + \int_{-\pi}^{0}(-2)\,\mathrm{d}t\right\} && \left[\text{Splitting } f(t)\right] \\
&= \frac{2}{2\pi}\left\{\left[t\right]_0^{\pi} + \left[-t\right]_{-\pi}^{0}\right\} && \left[\text{Integrating}\right] \\
&= \frac{1}{\pi}\left\{\left[\pi - 0\right] - \left[0 - (-\pi)\right]\right\} && \left[\text{Substituting for } t\right] \\
&= \frac{1}{\pi}\underbrace{\left\{\pi - \pi\right\}}_{=0} && \left[\text{Simplifying}\right] \\
A_0 &= 0
\end{aligned}$$

*How do we find the value of* $A_k$ *?*

Similar process to finding $A_0$ but this time we use

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt$$

Again we split $f(t)$ into 2 and $-2$.

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}(2)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-2)\cos(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{2}{\pi}\left\{\int_0^{\pi}\cos(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\cos(kt)\,\mathrm{d}t\right\} \\
&= \frac{2}{\pi}\left\{\left[\frac{\sin(kt)}{k}\right]_0^{\pi} - \left[\frac{\sin(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\cos(kt)\,dt = \frac{\sin(kt)}{k}\right] \\
&= \frac{2}{k\pi}\left\{\left[\sin(kt)\right]_0^{\pi} - \left[\sin(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } \frac{1}{k}\right] \\
&= \frac{2}{k\pi}\left\{\left[\sin(k\pi) - \sin(0)\right] - \left[\sin(0) - \sin(k(-\pi))\right]\right\} && \left[\text{Substituting the limits}\right] \\
&= \frac{2}{k\pi}\left\{\left[0 - 0\right] - \left[0 - 0\right]\right\} && \left[\text{Because } \sin(k\pi) = 0 \text{ and } \sin(k(-\pi)) = 0\right] \\
A_k &= 0
\end{aligned}$$

*How do we find $B_k$?*

Similarly by applying the formula

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$

Splitting $f(t)$ into 2 and $-2$ gives

$$\begin{aligned}
B_k &= \frac{1}{\pi}\left\{\int_0^{\pi}(2)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-2)\sin(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right]\\
&= \frac{2}{\pi}\left\{\int_0^{\pi}\sin(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\sin(kt)\,\mathrm{d}t\right\}\\
&= \frac{2}{\pi}\left\{\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} - \left[-\frac{\cos(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\sin(kt)\,dt = -\frac{\cos(kt)}{k}\right]\\
&= -\frac{2}{k\pi}\left\{\left[\cos(kt)\right]_0^{\pi} - \left[\cos(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } -\frac{1}{k}\right]\\
&= -\frac{2}{k\pi}\left\{\left[\cos(k\pi)-\cos(0)\right]-\left[\cos(0)-\cos(k(-\pi))\right]\right\} && \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right]\\
&= -\frac{2}{k\pi}\left\{\cos(k\pi)-1-1+\cos(-k\pi)\right\} && \left[\text{Remember } \cos(0)=1\right]\\
&= -\frac{2}{k\pi}\left\{\cos(k\pi)-2+\cos(k\pi)\right\} && \left[\text{Because } \cos(-k\pi)=\cos(k\pi)\right]\\
B_k &= -\frac{2}{k\pi}\left\{2\cos(k\pi)-2\right\} && \left[\text{Collecting } \cos(k\pi) \text{ terms}\right]
\end{aligned}$$

If $k$ is even then $\cos(k\pi)=1$, substituting this into the last line above gives:

$$B_k = -\frac{2}{k\pi}\left\{2-2\right\} = 0$$

If $k$ is odd then $\cos(k\pi)=-1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{2}{k\pi}\left\{-2-2\right\} = -\frac{2}{k\pi}\left\{-4\right\} = \frac{8}{k\pi}$$

Putting all these coefficients together we have

$$A_0 = 0$$
$$A_k = 0$$
$$B_k = \begin{cases} 0 & \text{if } k \text{ is even}\\ \dfrac{8}{k\pi} & \text{if } k \text{ is odd}\end{cases}$$

*What do we do next?*

Substitute these into

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$$

The *only* non-zero values are the odd sine terms:

$$f(t) = 0 + 0 + 0 + \ldots + \frac{8}{\pi}\sin(t) + 0 + \frac{8}{3\pi}\sin(3t) + 0 + \frac{8}{5\pi}\sin(5t) + \ldots$$

$$= \frac{8}{\pi}\sin(t) + \frac{8}{3\pi}\sin(3t) + \frac{8}{5\pi}\sin(5t) + \ldots \qquad \left[\text{Simplifying}\right]$$

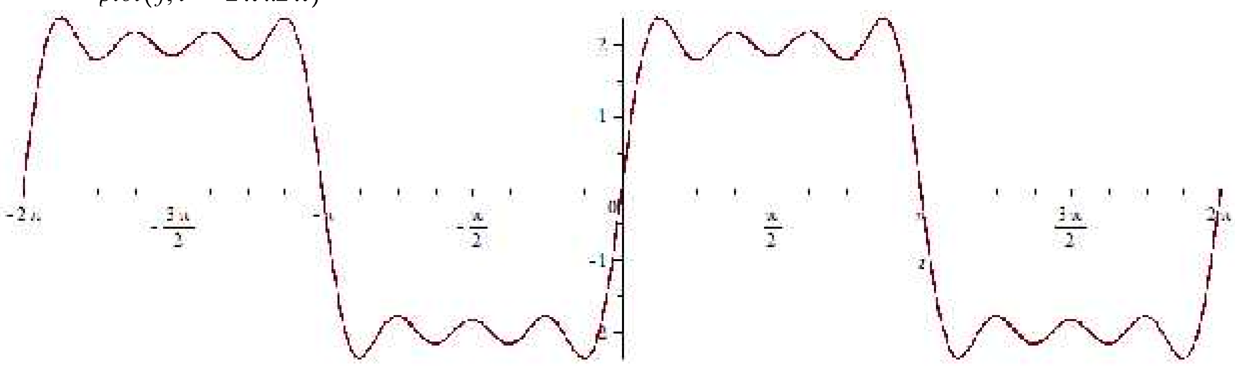
$$= \frac{8}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right] \qquad \left[\begin{array}{l}\text{Taking out a common}\\ \text{factor of } 8/\pi\end{array}\right]$$

Here is the Maple output for the graph and first four non-zero terms of the Fourier series of $f(t)$:

> $f := \frac{8}{\pi}\left(\sin(t) + \frac{\sin(3\,t)}{3} + \left(\frac{\sin(5\,t)}{5}\right) + \frac{\sin(7\,t)}{7}\right)$

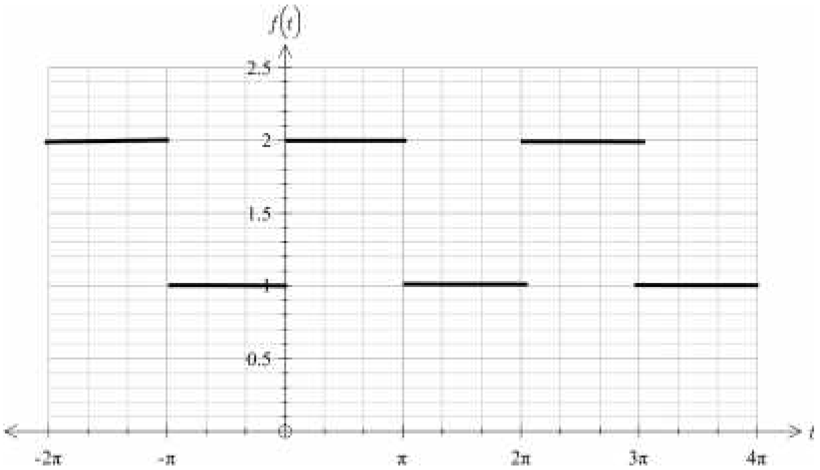
$$f := \frac{8\left(\sin(t) + \frac{1}{3}\sin(3\,t) + \frac{1}{5}\sin(5\,t) + \frac{1}{7}\sin(7\,t)\right)}{\pi}$$

> $plot(f, t = -2\,\pi\,..2\,\pi)$




This $f(t) = \frac{8}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$ is the Fourier series for $f(t)$ the given square waveform. Note how much easier it is in applying the linearity property of Fourier series which was covered in the main text. (We could have avoided all this integration.)

3. We are asked to sketch $f(t) = \begin{cases} 1 & -\pi < t < 0 \\ 2 & 0 < t < \pi \end{cases}$




(ii) We need to find the Fourier series of this function.

$f(t) = 2$ for $t$ between 0 and $\pi$ and $f(t) = 1$ for $t$ between $\pi$ and $2\pi$.

The average value of this function over a period of $2\pi$ is 3/2 so $A_0 = 3/2$.

For $A_k$ we use the following formula:

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt$$

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}(2)\cos(kt)\,\mathrm{d}t + \int_{\pi}^{2\pi}(1)\cos(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{1}{\pi}\left\{2\int_0^{\pi}\cos(kt)\,\mathrm{d}t + \int_{\pi}^{2\pi}\cos(kt)\,\mathrm{d}t\right\} \\
&= \frac{1}{\pi}\left\{2\left[\frac{\sin(kt)}{k}\right]_0^{\pi} + \left[\frac{\sin(kt)}{k}\right]_{\pi}^{2\pi}\right\} && \left[\text{Integrating by } \int\cos(kt)\,dt = \frac{\sin(kt)}{k}\right] \\
&= \frac{1}{k\pi}\left\{2\left[\sin(kt)\right]_0^{\pi} + \left[\sin(kt)\right]_{\pi}^{2\pi}\right\} && \left[\text{Taking out a factor of } \frac{2}{k}\right] \\
&= \frac{1}{k\pi}\left\{2\left[\sin(k\pi) - \sin(0)\right] + \left[\sin(2\pi k) - \sin(k\pi)\right]\right\} && \left[\text{Substituting the limits}\right] \\
&= \frac{1}{k\pi}\left\{2\left[0-0\right] + \left[0-0\right]\right\} \\
A_k &= 0
\end{aligned}$$

For the sine coefficients we use

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$

We have

$$
\begin{aligned}
B_k &= \frac{1}{\pi}\left\{\int_0^{\pi}(2)\sin(kt)\,\mathrm{d}t + \int_{\pi}^{2\pi}(1)\sin(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right]\\
&= \frac{1}{\pi}\left\{2\int_0^{\pi}\sin(kt)\,\mathrm{d}t + \int_{\pi}^{2\pi}\sin(kt)\,\mathrm{d}t\right\}\\
&= \frac{1}{\pi}\left\{2\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} + \left[-\frac{\cos(kt)}{k}\right]_{\pi}^{2\pi}\right\} && \left[\text{Integrating by } \int \sin(kt)dt = -\frac{\cos(kt)}{k}\right]\\
&= -\frac{1}{k\pi}\left\{2\left[\cos(kt)\right]_0^{\pi} + \left[\cos(kt)\right]_{\pi}^{2\pi}\right\} && \left[\text{Taking out a factor of } -\frac{1}{k}\right]\\
&= -\frac{1}{k\pi}\left\{2\left[\cos(k\pi)-\cos(0)\right] + \left[\cos(2\pi k)-\cos(k\pi)\right]\right\} && \left[\text{Substituting the limits}\right]\\
&= -\frac{1}{k\pi}\left\{2\cos(k\pi)-2+1-\cos(k\pi)\right\} && \left[\text{Remember } \cos(0)=\cos(2\pi k)=1\right]\\
&= -\frac{1}{k\pi}\left\{\cos(k\pi)-1\right\}
\end{aligned}
$$

If $k$ is even then $\cos(k\pi)=1$, substituting this into the last line above gives:

$$B_k = -\frac{1}{k\pi}\left\{\underbrace{\cos(k\pi)}_{=1}-1\right\} = 0 \text{ for even } k$$

If $k$ is odd then $\cos(k\pi)=-1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{1}{k\pi}\left\{\underbrace{\cos(k\pi)}_{=-1}-1\right\} = \frac{2}{k\pi}$$

Putting all these coefficients together we have

$$
\begin{aligned}
A_0 &= 3\\
A_k &= 0\\
B_k &= \begin{cases} 0 & \text{if } k \text{ is even}\\ \dfrac{2}{k\pi} & \text{if } k \text{ is odd}\end{cases}
\end{aligned}
$$

Substituting these into

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

We have

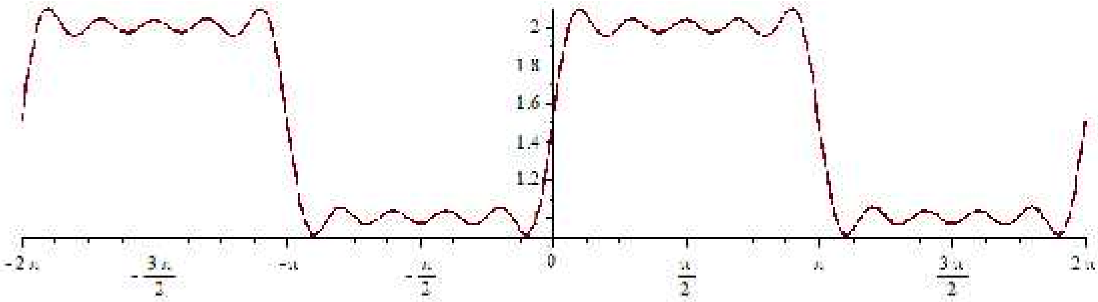
$$
\begin{aligned}
f(t) &= \frac{3}{2} + 0 + 0 + \ldots + \frac{2}{\pi}\sin(t) + 0 + \frac{2}{3\pi}\sin(3t) + 0 + \frac{2}{5\pi}\sin(5t) + \ldots\\
&= \frac{3}{2} + \frac{2}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right]
\end{aligned}
$$

Here is the Maple output:

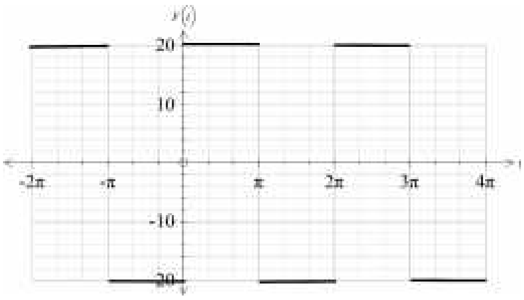
> $f := t \rightarrow \frac{3}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{\sin(3\,t)}{3} + \frac{\sin(5\,t)}{5} + \frac{\sin(7\,t)}{7} + \frac{\sin(9\,t)}{9}\right)$

$$f := t \rightarrow \frac{3}{2} + \frac{1}{\pi}\left(2\left(\sin(t) + \frac{1}{3}\sin(3\,t) + \frac{1}{5}\sin(5\,t) + \frac{1}{7}\sin(7\,t) + \frac{1}{9}\sin(9\,t)\right)\right)$$

> $plot(f, -2\,\pi..2\,\pi)$




4. (i) We first need to sketch $F(t) = \begin{cases} 20 & 0 < t < \pi \\ -20 & \pi < t < 2\pi \end{cases}$




We have $F(t) = 20$ for $t$ between 0 and $\pi$ and $f(t) = -20$ for $t$ between 0 and $-\pi$. Clearly the average value of the given function over a period of $2\pi$ is 0. Here is the derivation of this using integration:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\left\{\int_0^{\pi}(20)\,\mathrm{d}t + \int_{-\pi}^{0}(-20)\,\mathrm{d}t\right\} && \left[\text{Splitting } f(t)\right] \\ &= \frac{20}{2\pi}\left\{\left[t\right]_0^{\pi} + \left[-t\right]_{-\pi}^{0}\right\} && \left[\text{Integrating and taking out 20}\right] \\ &= \frac{10}{\pi}\left\{\left[\pi - 0\right] - \left[0 - (-\pi)\right]\right\} && \left[\text{Substituting for } t\right] \\ &= \frac{10}{\pi}\underbrace{\left\{\pi - \pi\right\}}_{=0} && \left[\text{Simplifying}\right] \\ A_0 &= 0 \end{aligned}$$

*How do we find the value of* $A_k$ *?*

Similar process to finding $A_0$ but this time we use

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt$$

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}(20)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-20)\cos(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{20}{\pi}\left\{\int_0^{\pi}\cos(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\cos(kt)\,\mathrm{d}t\right\} \\
&= \frac{20}{\pi}\left\{\left[\frac{\sin(kt)}{k}\right]_0^{\pi} - \left[\frac{\sin(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\cos(kt)\,dt = \frac{\sin(kt)}{k}\right] \\
&= \frac{20}{k\pi}\left\{\left[\sin(kt)\right]_0^{\pi} - \left[\sin(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } \frac{1}{k}\right] \\
&= \frac{20}{k\pi}\left\{\left[\sin(k\pi)-\sin(0)\right] - \left[\sin(0)-\sin(k(-\pi))\right]\right\} && \left[\text{Substituting the limits}\right] \\
&= \frac{20}{k\pi}\left\{\left[0-0\right]-\left[0-0\right]\right\} \\
A_k &= 0
\end{aligned}$$

*How do we find* $B_k$ *?*

Similarly by applying the formula

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$

Splitting $f(t)$ into 20 and $-20$ gives

$$\begin{aligned}
B_k &= \frac{1}{\pi}\left\{\int_0^{\pi}(20)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-20)\sin(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{20}{\pi}\left\{\int_0^{\pi}\sin(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\sin(kt)\,\mathrm{d}t\right\} \\
&= \frac{20}{\pi}\left\{\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} - \left[-\frac{\cos(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\sin(kt)\,dt = -\frac{\cos(kt)}{k}\right] \\
&= -\frac{20}{k\pi}\left\{\left[\cos(kt)\right]_0^{\pi} - \left[\cos(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } -\frac{1}{k}\right] \\
&= -\frac{20}{k\pi}\left\{\left[\cos(k\pi)-\cos(0)\right] - \left[\cos(0)-\cos(k(-\pi))\right]\right\} && \left[\text{Substituting the limits}\right] \\
&= -\frac{20}{k\pi}\left\{\cos(k\pi)-1-1+\cos(-k\pi)\right\} && \left[\text{Remember } \cos(0)=1\right] \\
&= -\frac{20}{k\pi}\left\{\cos(k\pi)-2+\cos(k\pi)\right\} \\
B_k &= -\frac{20}{k\pi}\left\{2\cos(k\pi)-2\right\} = -\frac{40}{k\pi}\left\{\cos(k\pi)-1\right\} && \left[\text{Collecting } \cos(k\pi) \text{ terms}\right]
\end{aligned}$$

If $k$ is even then $\cos(k\pi) = 1$, substituting this into the last line above gives:

$$B_k = -\frac{40}{k\pi}\{1-1\} = 0$$

If $k$ is odd then $\cos(k\pi) = -1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{40}{k\pi}\{-1-1\} = -\frac{40}{k\pi}\{-2\} = \frac{80}{k\pi}$$

Putting all these coefficients together we have

$$A_0 = 0$$
$$A_k = 0$$
$$B_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ \dfrac{80}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Substituting these into

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

The *only* non-zero values are the odd sine terms:

$$f(t) = 0 + 0 + 0 + \ldots + \frac{80}{\pi}\sin(t) + 0 + \frac{80}{3\pi}\sin(3t) + 0 + \frac{80}{5\pi}\sin(5t) + \ldots$$
$$= \frac{80}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right] \qquad \left[\begin{array}{l}\text{Taking out a common} \\ \text{factor of } 80/\pi\end{array}\right]$$

This is the Fourier series for $F(t)$ which was given by

$$F(t) = \begin{cases} 20 & 0 < t < \pi \\ -20 & \pi < t < 2\pi \end{cases}$$

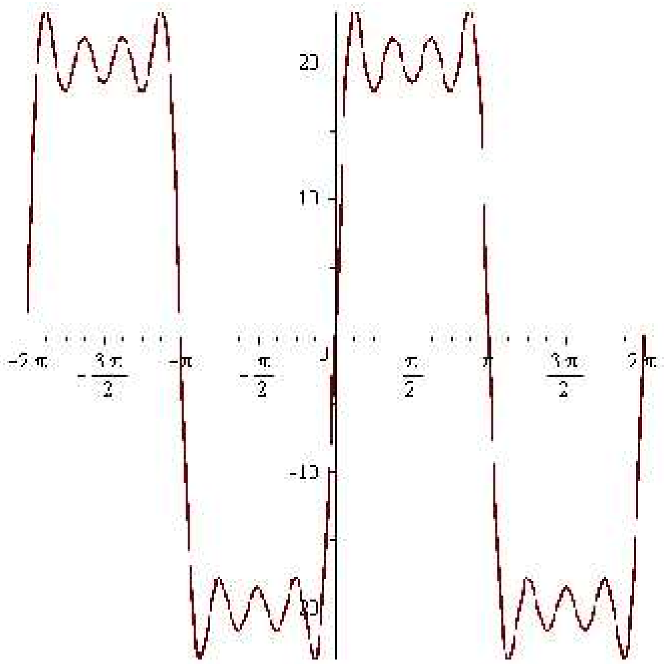
This means that we can write $f(t)$ as an infinite sum of odd sine terms.

Here is the Maple output:

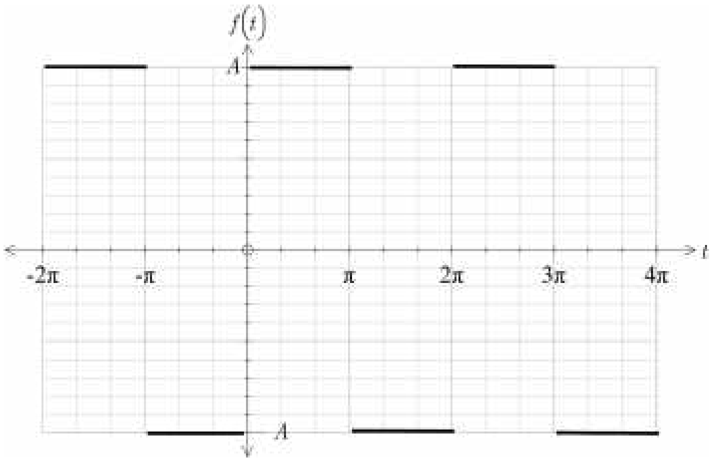
> $f := t \to \frac{80}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7}\right)$

$$f := t \to \frac{80\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t)\right)}{\pi}$$

> $plot(f, -2\pi..2\pi)$



5. We need to find the Fourier series of




By examining this graph we have that the average value of the given function over a period of $2\pi$ is 0. Therefore $A_0 = 0$. If you used the integration formula for $A_0$ then you would have to carry out the following calculation:

$$
\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}(A)\,\mathrm{d}t + \int_{-\pi}^{0}(-A)\,\mathrm{d}t\right\} && \left[\text{Splitting } f(t)\right] \\
&= \frac{A}{2\pi}\left\{\left[t\right]_0^{\pi} + \left[-t\right]_{-\pi}^{0}\right\} && \left[\text{Integrating and taking out } A\right] \\
&= \frac{A}{\pi}\left\{\left[\pi - 0\right] - \left[0 - (-\pi)\right]\right\} && \left[\text{Substituting for } t\right] \\
&= \frac{A}{\pi}\underbrace{\left\{\pi - \pi\right\}}_{=0} && \left[\text{Simplifying}\right] \\
A_0 &= 0
\end{aligned}
$$

Similarly we have

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}(A)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0} A\cos(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{A}{\pi}\left\{\int_0^{\pi}\cos(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\cos(kt)\,\mathrm{d}t\right\} \\
&= \frac{A}{\pi}\left\{\left[\frac{\sin(kt)}{k}\right]_0^{\pi} - \left[\frac{\sin(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\cos(kt)\,dt = \frac{\sin(kt)}{k}\right] \\
&= \frac{A}{k\pi}\left\{\left[\sin(kt)\right]_0^{\pi} - \left[\sin(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } \frac{1}{k}\right] \\
&= \frac{A}{k\pi}\left\{\left[\sin(k\pi)-\sin(0)\right] - \left[\sin(0)-\sin(k(-\pi))\right]\right\} && \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right] \\
&= \frac{A}{k\pi}\left\{\left[0-0\right]-\left[0-0\right]\right\} \\
A_k &= 0
\end{aligned}$$

We also have

$$\begin{aligned}
B_k &= \frac{1}{\pi}\left\{\int_0^{\pi}(A)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-A)\sin(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{A}{\pi}\left\{\int_0^{\pi}\sin(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\sin(kt)\,\mathrm{d}t\right\} \\
&= \frac{A}{\pi}\left\{\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} - \left[-\frac{\cos(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\sin(kt)\,dt = -\frac{\cos(kt)}{k}\right] \\
&= -\frac{A}{k\pi}\left\{\left[\cos(kt)\right]_0^{\pi} - \left[\cos(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } -\frac{1}{k}\right] \\
&= -\frac{A}{k\pi}\left\{\left[\cos(k\pi)-\cos(0)\right] - \left[\cos(0)-\cos(k(-\pi))\right]\right\} && \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right] \\
&= -\frac{A}{k\pi}\left\{\cos(k\pi)-1-1+\cos(-k\pi)\right\} && \left[\text{Remember } \cos(0)=1\right] \\
&= -\frac{A}{k\pi}\left\{\cos(k\pi)-2+\cos(k\pi)\right\} \\
B_k &= -\frac{A}{k\pi}\left\{2\cos(k\pi)-2\right\} = -\frac{2A}{k\pi}\left\{\cos(k\pi)-1\right\} && \left[\text{Collecting } \cos(k\pi) \text{ terms}\right]
\end{aligned}$$

If $k$ is even then $\cos(k\pi)=1$, substituting this into the last line above gives:

$$B_k = -\frac{2A}{k\pi}\left\{1-1\right\} = 0$$

If $k$ is odd then $\cos(k\pi)=-1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{2A}{k\pi}\left\{-1-1\right\} = -\frac{2A}{k\pi}\left\{-2\right\} = \frac{4A}{k\pi}$$

Putting all these coefficients together we have

$$A_0 = 0$$
$$A_k = 0$$
$$B_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ \dfrac{4A}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Substituting these into

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

The *only* non-zero values are the odd sine terms:

$$f(t) = 0 + 0 + 0 + \ldots + \frac{4A}{\pi}\sin(t) + 0 + \frac{4A}{3\pi}\sin(3t) + 0 + \frac{4A}{5\pi}\sin(5t) + \ldots$$
$$= \frac{4A}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right] \qquad \left[\text{Taking out a common factor of } 4A/\pi\right]$$

6. For this question we use the above derived formula

$$f(t) = \frac{4A}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

(a) *What is the amplitude of the given function?*

$$f(t) = \begin{cases} 2 & 0 < t < \pi \\ -2 & \pi < t < 2\pi \end{cases}$$

It is 2 so substituting $A = 2$ into the above derivation gives

$$f(t) = \frac{8}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

Note that this the same function given in question 1(b) and of course we have the same answer.

(b) Similarly for

$$f(t) = \begin{cases} -5 & -\pi < t < 0 \\ 5 & 0 < t < \pi \end{cases}$$

The amplitude is 5 so substituting $A = 5$ gives the Fourier series

$$f(t) = \frac{20}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

7. This time we have

$$f(t)=\begin{cases}A & 0<t<\pi\\ 0 & -\pi<t<0\end{cases}.$$

The constant term $A_0$ is given by

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t\\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}(A)\,\mathrm{d}t+\int_{-\pi}^{0}(0)\,\mathrm{d}t\right\} && \left[\text{Splitting } f(t)\right]\\
&= \frac{A}{2\pi}\left[t\right]_0^{\pi} && \left[\text{Integrating}\right]\\
&= \frac{A}{2\pi}\left[\pi-0\right] && \left[\text{Substituting for } t\right]\\
&= \frac{A}{2\not\pi}\not\pi && \left[\text{Simplifying}\right]\\
A_0 &= \frac{A}{2}
\end{aligned}$$

The cosine coefficients are given by

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^{\pi}(A)\cos(kt)\,\mathrm{d}t && \left[\text{Replacing } f(t)\right]\\
&= \frac{A}{\pi}\left\{\left[\frac{\sin(kt)}{k}\right]_0^{\pi}\right\} && \left[\text{Taking out } A \text{ and } \int\cos(kt)\,dt=\frac{\sin(kt)}{k}\right]\\
&= \frac{A}{k\pi}\left[\sin(kt)\right]_0^{\pi} && \left[\text{Taking out a factor of } \frac{1}{k}\right]\\
&= \frac{A}{k\pi}\left[\sin(k\pi)-\sin(0)\right] && \left[\begin{array}{l}\text{Substituting}\\ \text{the limits}\end{array}\right]\\
&= \frac{A}{k\pi}\left\{\left[0-0\right]\right\}=0 && \left[\text{Because } \sin(k\pi)=0\right]
\end{aligned}$$

We have $A_k=0$.

We also need to find the sine coefficients:

$$\begin{aligned}
B_k &= \frac{1}{\pi}\int_0^{\pi}(A)\sin(kt)\,\mathrm{d}t && \left[\text{Replacing } f(t)\right] \\
&= \frac{A}{\pi}\int_0^{\pi}\sin(kt)\,\mathrm{d}t && \left[\text{Taking out } A\right] \\
&= \frac{A}{\pi}\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} && \left[\text{Integrating by } \int\sin(kt)\,dt = -\frac{\cos(kt)}{k}\right] \\
&= -\frac{A}{k\pi}\left[\cos(kt)\right]_0^{\pi} && \left[\text{Taking out a factor of } -\frac{1}{k}\right] \\
&= -\frac{A}{k\pi}\left[\cos(k\pi)-\cos(0)\right] && \left[\begin{array}{l}\text{Substituting} \\ \text{the limits}\end{array}\right] \\
&= -\frac{A}{k\pi}\left[\cos(k\pi)-1\right] && \left[\text{Remember } \cos(0)=1\right]
\end{aligned}$$

If $k$ is even then $\cos(k\pi)=1$, substituting this into the last line above gives:

$$B_k = -\frac{A}{k\pi}\left[\cos(k\pi)-1\right] = -\frac{A}{k\pi}\left[1-1\right] = 0$$

If $k$ is odd then $\cos(k\pi)=-1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{A}{k\pi}\left[\cos(k\pi)-1\right] = -\frac{A}{k\pi}\left[-1-1\right] = -\frac{A}{k\pi}\left[-2\right] = \frac{2A}{k\pi}$$

Putting all these coefficients together we have

$$\begin{aligned}
A_0 &= A/2 \\
A_k &= 0 \\
B_k &= \begin{cases} 0 & \text{if } k \text{ is even} \\ \dfrac{2A}{k\pi} & \text{if } k \text{ is odd} \end{cases}
\end{aligned}$$

Substituting these into

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

Gives

$$\begin{aligned}
f(t) &= \frac{A}{2} + 0 + 0 + \ldots + \frac{2A}{\pi}\sin(t) + 0 + \frac{2A}{3\pi}\sin(3t) + 0 + \frac{2A}{5\pi}\sin(5t) + \ldots \\
&= \frac{A}{2} + \frac{2A}{\pi}\sin(t) + \frac{2A}{3\pi}\sin(3t) + \frac{2A}{5\pi}\sin(5t) + \ldots && \left[\text{Simplifying}\right] \\
&= \frac{A}{2} + \frac{2A}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right] && \left[\begin{array}{l}\text{Taking out a common} \\ \text{factor of } 2A/\pi\end{array}\right]
\end{aligned}$$

8. We are asked to show $\displaystyle\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

Case I $m \neq n$

To deal with this integrand we use the following trigonometric identity:

$$\cos\left(A\right)\cos\left(B\right) = \frac{1}{2}\left[\cos\left(A+B\right) + \cos\left(A-B\right)\right]$$

Applying this to the integrand gives

$$\cos\left(mx\right)\cos\left(nx\right) = \frac{1}{2}\left[\cos\left(\left(m+n\right)x\right) + \cos\left(\left(m-n\right)x\right)\right]$$

Evaluating the given integral

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos\left(mx\right)\cos\left(nx\right)\,\mathrm{d}x &= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos\left(\left(m+n\right)x\right)\,\mathrm{d}x + \int_{-\pi}^{\pi} \cos\left(\left(m-n\right)x\right)\,\mathrm{d}x\right] \\
&= \frac{1}{2}\left[\left[\frac{\sin\left(\left(m+n\right)x\right)}{m+n}\right]_{-\pi}^{\pi} + \left[\frac{\sin\left(\left(m-n\right)x\right)}{m-n}\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2}\left[\frac{1}{m+n}\left[\sin\left(\left(m+n\right)\pi\right) - \sin\left(-\left(m+n\right)\pi\right)\right] + \right. \\
&\qquad \left. \frac{1}{m-n}\left[\sin\left(\left(m-n\right)\pi\right) - \sin\left(-\left(m-n\right)\pi\right)\right]\right]
\end{aligned}$$

In the last line we have integer multiples of $\pi$ in the argument of the sine function so these are all zero because $\sin\left(k\pi\right) = 0$ for any integer $k$.

Case II $m = n$

In this case we have $m = n$ so the integrand is given by

$$\cos\left(mx\right)\cos\left(mx\right) = \cos^2\left(mx\right)$$

We need to integrate this $\cos^2\left(mx\right)$. Again we find the appropriate identity:

$$\cos^2\left(A\right) = \frac{1}{2}\left[\cos\left(2A\right) + 1\right]$$

We have $\cos^2\left(mx\right) = \frac{1}{2}\left[\cos\left(2mx\right) + 1\right]$. We have

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2\left(mx\right)\,\mathrm{d}x &= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos\left(2mx\right)\,\mathrm{d}x + \int_{-\pi}^{\pi} 1\,\mathrm{d}x\right] \\
&= \frac{1}{2}\left[\left[\frac{\sin\left(2mx\right)}{2m}\right]_{-\pi}^{\pi} + \left[x\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2}\left[\frac{1}{2m}\left[\sin\left(2m\pi\right) - \sin\left(-2m\pi\right)\right] + \left[\pi - \left(-\pi\right)\right]\right] \\
&= \frac{1}{2}\left[\frac{1}{2m}\left[0-0\right] + 2\pi\right] = \pi \qquad \left[\begin{array}{l}\text{Because} \\ \sin\left(2m\pi\right) = \sin\left(-2m\pi\right) = 0\end{array}\right]
\end{aligned}$$

Hence we have our given result.

9. We are asked to show $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

Case I $m \neq n$

To deal with this integrand we use the following trigonometric identity:

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Applying this to the integrand gives

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos\left((m-n)x\right) - \cos\left((m+n)x\right)\right]$$

Evaluating the given integral:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x &= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos\left((m-n)x\right)\,\mathrm{d}x - \int_{-\pi}^{\pi} \cos\left((m+n)x\right)\,\mathrm{d}x\right] \\
&= \frac{1}{2}\left[\left[\frac{\sin\left((m-n)x\right)}{m-n}\right]_{-\pi}^{\pi} - \left[\frac{\sin\left((m+n)x\right)}{m+n}\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2}\left[\frac{1}{m-n}\left[\sin\left((m-n)\pi\right) - \sin\left(-(m-n)\pi\right)\right] - \frac{1}{m+n}\left[\sin\left((m+n)\pi\right) - \sin\left(-(m+n)\pi\right)\right]\right] \\
&= \frac{1}{2}\left[0\right] \qquad \left[\text{Because } \sin(k\pi) = 0\right]
\end{aligned}$$

For $m \neq n$ we have our required result $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = 0$.

Case II $m = n$

In this case we have $m = n$ so the integrand is given by

$$\sin(mx)\sin(mx) = \sin^2(mx)$$

We need to integrate this $\sin^2(mx)$. Again we find the appropriate identity:

$$\sin^2(A) = \frac{1}{2}\left[1 - \cos(2A)\right]$$

We have $\sin^2(mx) = \frac{1}{2}\left[1 - \cos(2mx)\right]$. We have

$$\int_{-\pi}^{\pi} \sin^2\left(mx\right)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-\pi}^{\pi} 1\,\mathrm{d}x - \int_{-\pi}^{\pi}\cos\left(2mx\right)\,\mathrm{d}x\right]$$
$$= \frac{1}{2}\left[\left[x\right]_{-\pi}^{\pi} - \left[\frac{\sin\left(2mx\right)}{2m}\right]_{-\pi}^{\pi}\right]$$
$$= \frac{1}{2}\left[\left[\pi - \left(-\pi\right)\right] - \frac{1}{2m}\left[\sin\left(2m\pi\right) - \sin\left(-2m\pi\right)\right]\right]$$
$$= \frac{1}{2}\left[2\pi - \frac{1}{2m}\left[0-0\right]\right] = \pi$$

We also have $\int_{-\pi}^{\pi}\sin\left(mx\right)\sin\left(nx\right)\,\mathrm{d}x = \pi$ if $m = n$ .

10. We have to show that $\int_{-\pi}^{\pi}\sin\left(mx\right)\cos\left(nx\right)\,\mathrm{d}x = 0$ for any integers $m$ and $n$.

To deal with this integrand we use the following trigonometric identity:

$$\sin\left(A\right)\cos\left(B\right) = \frac{1}{2}\left[\sin\left(A+B\right) + \sin\left(A-B\right)\right]$$

Applying this to the integrand gives

$$\sin\left(mx\right)\cos\left(nx\right) = \frac{1}{2}\left[\sin\left(\left(m+n\right)x\right) + \sin\left[\left(m-n\right)x\right]\right]$$

Evaluating the given integral:

$$\int_{-\pi}^{\pi}\sin\left(mx\right)\cos\left(nx\right)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-\pi}^{\pi}\sin\left(\left(m+n\right)x\right)\,\mathrm{d}x + \int_{-\pi}^{\pi}\sin\left(\left(m-n\right)x\right)\,\mathrm{d}x\right]$$
$$= -\frac{1}{2}\left\{\left[\frac{\cos\left(\left(m+n\right)x\right)}{m+n}\right]_{-\pi}^{\pi} + \left[\frac{\cos\left(\left(m-n\right)x\right)}{m-n}\right]_{-\pi}^{\pi}\right\}$$
$$= -\frac{1}{2}\left[\frac{1}{m+n}\left[\cos\left(\left(m+n\right)\pi\right) - \cos\left(-\left(m+n\right)\pi\right)\right] + \frac{1}{m-n}\left[\cos\left(\left(m-n\right)\pi\right) - \cos\left(-\left(m-n\right)\pi\right)\right]\right]$$
$$= \frac{1}{2}\left[\frac{1}{m+n}\left[\cos\left(\left(m+n\right)\pi\right) - \cos\left(\left(m+n\right)\pi\right)\right] + \frac{1}{m-n}\left[\cos\left(\left(m-n\right)\pi\right) - \cos\left(\left(m-n\right)\pi\right)\right]\right] \quad \left[\text{Because } \cos\left(-x\right) = \cos\left(x\right)\right]$$
$$= \frac{1}{2}\left[\frac{1}{m+n}\left[0\right] + \frac{1}{m-n}\left[0\right]\right] = 0$$

For any integers $m$ and $n$ we have our required result

$$\int_{-\pi}^{\pi}\sin\left(mx\right)\cos\left(nx\right)\,\mathrm{d}x = 0$$