

Chapter 3 Functions

SECTION A Introduction to Functions

By the end of this section you will be able to

- understand what is meant by a function
- understand the terms associated with functions such as range, domain, codomain, image etc
- determine the range, domain, codomain and images of a function

A1 Definition of a function

What does the term function mean in everyday language?

The purpose of an object or a person. For example switching on a light or turning off the television. In mathematics a function transforms elements from one set to another set. *Can you think of any mathematical examples of functions?*

Adding 4, taking square root of, taking the logarithm of etc All these transform a number (argument) into another number.

For example adding 4 can be represented by:

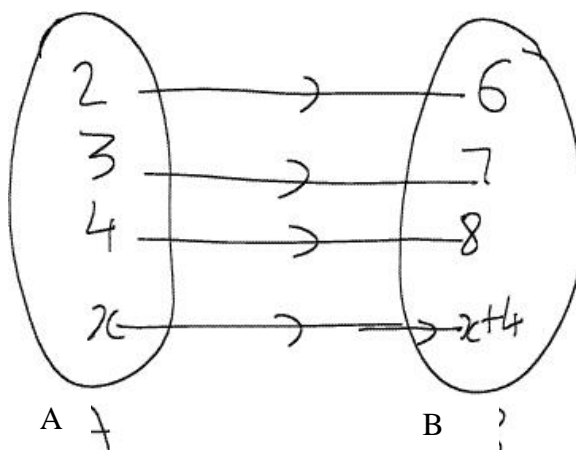


Fig 1

The function adding 4 takes x to $x+4$ which can be denoted by $x \rightarrow x+4$. The function adding 4 is denoted by

$$f(x) = x + 4 \text{ or } f : x \rightarrow x + 4$$

What does $f(x) = x + 4$ mean?

It means that the function, f , takes x to $x+4$. The aim of the function is to add 4 to any given value.

The function $f(x)$ is pronounced ' f of x '. The set A of Fig 1 where we start from is called the **domain** of the function. The set B of Fig 1 where we arrive is called the **codomain**. We call x the argument of the function f .

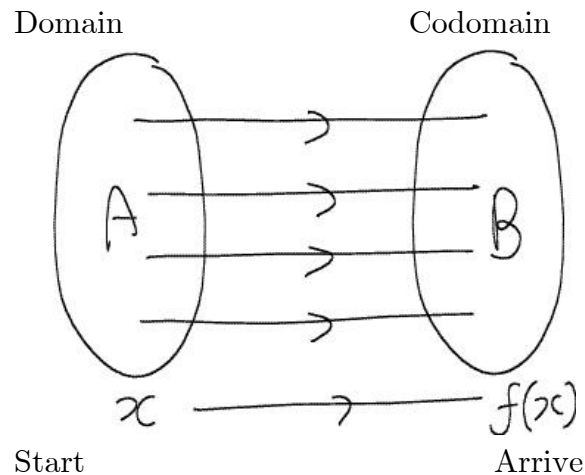


Fig 2

The above function f from domain, A , to the codomain, B , is normally denoted by $f : A \rightarrow B$. The domain and codomain do not just consist of a few selected points shown in the above figures but could be all the real numbers.

The variable x is the independent variable and $f(x)$ the dependent variable because $f(x)$ depends on x . *What does the term function mean in mathematics?*

A function is a correspondence between one set of elements to another set. Before we given a formal definition of a function we look at the history of functions.

The idea of dealing with functions in mathematics has been known since the Babylonian times but a thorough definition of a function was first given by the Swiss mathematician Euler.



Fig 3 Euler
1707 to 1783

Leonhard Euler was born in Basel, Switzerland in 1707. He studied at the local University of Basel and finished his studies in 1726. In the following year he moved to St Petersburg Academy of Sciences in Russia and became a professor of physics in 1730. He was promoted to chair of mathematics 3 years later. In 1741 he moved to Berlin Academy, Germany and lived there for 25 years. In 1766 he moved back to St Petersburg and by this time he was totally blind. However he continued to work in mathematics producing a vast number of articles even at this critical time of his life. He was the first to think of sin and cos as functions. The exponential notation e^x was first introduced by Euler.

However our modern definition of function was first stated by the Belgium mathematician Johann Dirichlet in 1837. (Profile of Dirichlet is given at the end of this section).

Definition (3.1)

A **function** from the domain A to a codomain B is a **rule** that assigns **every** element in the domain, A , **only one** element in the codomain, B .

Every element from the start must arrive at a **single** destination.

Example 1

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Let $f : A \rightarrow B$ be given by

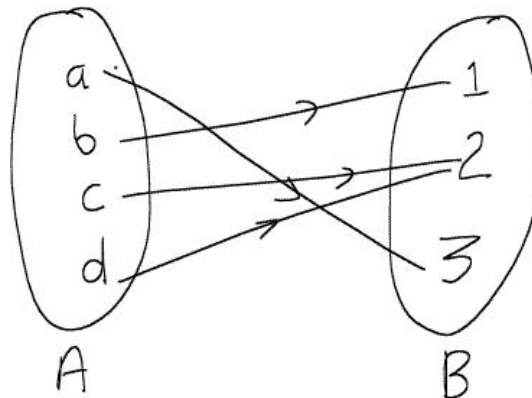


Fig 4

Is f a function? If not, explain why not.

Solution

Yes f is a function because every element in the domain, A , is assigned only one element in the codomain, B . Note that the elements c and d are assigned to the same element, 2.

Example 2

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Let $f : A \rightarrow B$ be given by

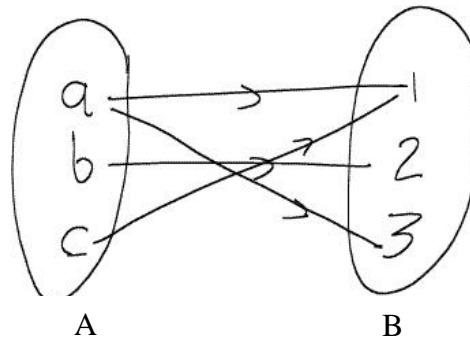


Fig 5

Is f a function? If not, explain why not.

Solution

No f is **not** a function because the element a is assigned two elements, 1 and 3.

That is the element a arrives at two different destinations, 1 and 3. Remember the definition of a function says every element in A is assigned **only one** element in B .

A function cannot assign any element in the domain to two different destinations.

Each element in the domain, A , must have a unique arrival point in the codomain, B , for f to be a function. Since a arrives at 1 and 3 therefore f **cannot** be a function.

Example 3

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 4\}$. Let $f : A \rightarrow B$ be given by

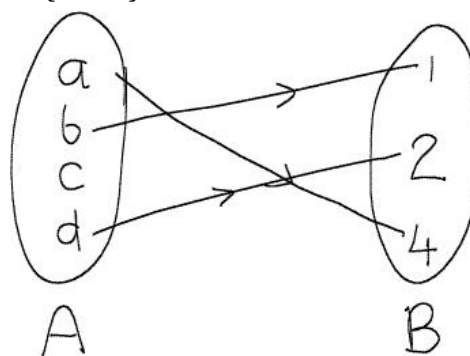


Fig 6

Is f a function? If not, explain why not.

Solution

No f is **not** a function because the element c is **not** assigned an element in the codomain, B . Remember f is a function only if **every** element in the domain, A , is

assigned a unique element in the codomain, B . Since f does not assign c therefore it cannot be a function.

Example 4

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. Let $f : A \rightarrow B$ be given by

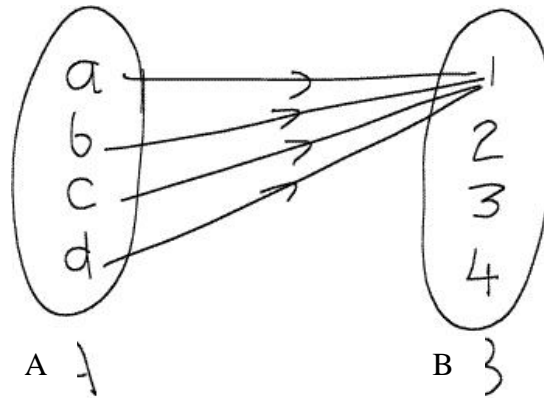


Fig 7

Is f a function? If not, explain why not.

Solution

Yes f is a function because every element in the domain, A , arrives at a single destination. Note that the arrival point in the codomain may be the same for all the elements in A but they all have a single destination. For example the function f takes b to 1 only. If the function f also took b to 2 say then f would **not** be a function because then f takes b to two different destinations, 1 and 2.

A2 Range of a Function

Definition (1.2)

Let f be a function. Then the set of values of $f(x)$ is called the **range** of the function f .

Is the range in the domain or codomain?

Since the range is the set of values $f(x)$ therefore it lies in the codomain. Actually it is a **subset** of the codomain. This means that the range lies within the codomain.

What is the range of function f given in example 4?

The range of f is the set $\{1\}$. Note that $\{1\}$ is a subset of the codomain, $B = \{1, 2, 3, 4\}$.

What is the range of function f given in example 1?

The range of f in example 1 is the set $\{1, 2, 3\}$. Note that $\{1, 2, 3\}$ is equal to the codomain, $B = \{1, 2, 3\}$.

The range of a function f maybe a proper subset of the codomain or equal to the codomain. That is the range of f is contained in the codomain.

What is the difference between the range and codomain?

Consider Example 4 above. The codomain is the set of 4 elements, $B = \{1, 2, 3, 4\}$ but the range has only a single element $\{1\}$. Most students are confused between the terms range and codomain and think they are equal. In some cases they maybe

equal but in many they are not.

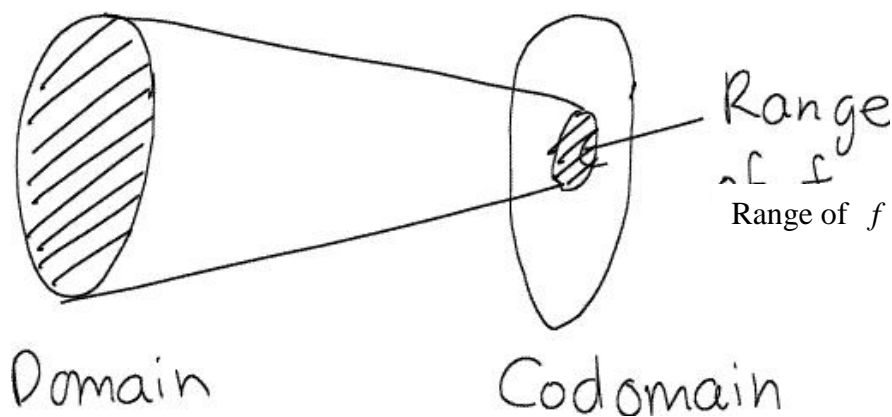


Fig 8

If a function f assigns an element x to $f(x)$ then $f(x)$ is called the image of x .

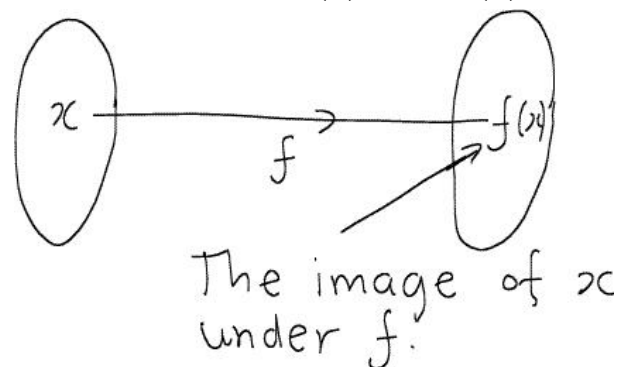


Figure 9

Let $f : A \rightarrow B$ then the set of all the images $f(x)$, where $x \in A$, is the range of f .

A3 Other functions

Usually the domain, A , and codomain, B , are one of the following sets

\mathbb{R} , \mathbb{Z} , \mathbb{C} , \mathbb{N} . That is the domain and codomain are normally the sets of real numbers, integers, complex numbers or natural numbers.

Example 5

(a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = x^2$$

Determine the images of

- (i) 2 (ii) -5 (iii) π (iv) 4^2 (v) $\frac{1}{f}$ (vi) $(a+b)$

What is the domain, codomain and range of f ?

Solution

(i) The image of 2 under f is given by $f(2)$. *What is $f(2)$ equal to?*

We substitute $x=2$ into the given function $f(x)=x^2$:

$$f(2) = 2^2 = 4$$

Hence the image of 2 under the function $f(x)=x^2$ is 4.

(ii) The image of -5 under f is given by $f(-5)$. *What is $f(-5)$ equal to?*

We substitute $x=-5$ into the given function $f(x)=x^2$:

$$f(-5) = (-5)^2 = 25$$

Hence the image of -5 is 25.

(iii) Similarly the image of f is $f(f)=f^2$.

(iv) *What is the image of 4^2 ?*

Substituting $x=4^2$ into the given function $f(x)=x^2$:

$$f(4^2) = (4^2)^2 = (16)^2 = 256$$

Therefore the image of 4^2 is 256.

(v) *How do we find the image of $\frac{1}{f}$ under the function $f(x)=x^2$?*

By substituting $x=\frac{1}{f}$ into the given function $f(x)=x^2$:

$$f\left(\frac{1}{f}\right) = \left(\frac{1}{f}\right)^2 = \frac{1^2}{f^2} = \frac{1}{f^2}$$

Hence the image of $\frac{1}{f}$ under the given function f is $\frac{1}{f^2}$.

(vi) *How do we find the image of $a+b$?*

This is done in the same manner by substituting $x=a+b$:

$$\begin{aligned} f(a+b) &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

The image of $a+b$ under the function f is $a^2 + 2ab + b^2$.

What is the domain and codomain of the function f ?

Remember for the function $f:A \rightarrow B$ the domain was A and codomain was B .

Since we are given $f:\mathbb{R} \rightarrow \mathbb{R}$ therefore the domain and codomain are both \mathbb{R} .

What is the range of f ?

The range is the set of **all** the elements $f(x)$. *What values can the function f take?*

Since $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = x^2$ therefore f can only take positive real values plus zero. (Remember x^2 cannot be negative). The range of f is the set of all positive real numbers plus zero and this is given by the set $\mathbb{R}^+ \cup \{0\}$.

Note that the codomain is \mathbb{R} but the range, $\mathbb{R}^+ \cup \{0\}$, is a subset of this codomain.

Example 6

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \sqrt{x} \quad [\text{Positive Square Root}]$$

Is f a function? If not why not?

How can we redefine f so that it is a function?

Solution

f is not a function because -1 is in the domain but $f(-1)$ is not defined. *Why not?*

Because $f(-1) = \sqrt{-1}$ and there is no real number which has this value, $\sqrt{-1}$. *For what values is f not defined?*

In fact f is **not** defined for any of the negative real numbers because

$$\sqrt{\text{Negative Real Number}} \neq (\text{Real Number})$$

How can one redefine f so that it is a function?

We redefine the domain so that it can only have positive real numbers plus zero.

How do we write the set of all positive real numbers plus zero?

$$\mathbb{R}^+ \cup \{0\}$$

Hence the domain is $\mathbb{R}^+ \cup \{0\}$ and we write this function as

$$g: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R} \text{ given by } g(x) = \sqrt{x}.$$

Note that in this example we are only considering the positive square root.

Example 6 shows that we can redefine the domain (or codomain) to make f a function. Also note that we have to relabel f in Example 6 even if we only change the domain (or codomain). Hence in the above example we have labelled the function as g .

Example 7

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$(a) \quad f(x) = x^3 \qquad (b) \quad f(x) = \sin(x) \qquad (c) \quad f(x) = e^x$$

We will formally define the trigonometric and exponential functions later in this chapter.

Determine the range of each of these functions.

Solution

In each case the domain is the set of all the real numbers, \mathbb{R} . *What does this mean?*

This means that the variable x in the function $f(x)$ can only take real values.

(a) The domain of $f(x) = x^3$ is \mathbb{R} because we are given $f : \mathbb{R} \rightarrow \mathbb{R}$. Therefore x can take any real values. *What values can $f(x) = x^3$ take?*

Recall that x^3 can be any real number for any real x , therefore the range of f is the real numbers that is the set of all the real numbers which is \mathbb{R} . In this case the codomain and the range are the same and equal to \mathbb{R} .

(b) *What values can $f(x) = \sin(x)$ take for any real x ?*

Remember the sine of a real number can only have values between -1 and $+1$ (inclusive). Therefore the range of f is the set of all the real numbers between -1 to $+1$ which is the interval $[-1, 1]$. In this case the codomain is all the real numbers, \mathbb{R} , but the range is the interval $[-1, 1]$.

(c) *What values can $f(x) = e^x$ take for any real x ?*

Recall that the exponential function, $f(x) = e^x$, is never zero or negative.

Remember the graph of the exponential function is always above the horizontal axis. *What is the range of $f(x) = e^x$?*

Clearly $f(x) = e^x$ can only take positive real values. Therefore the range of f is \mathbb{R}^+ that is the positive real numbers. Clearly the set of positive real numbers is a subset of all the real numbers. Hence the range of the given function, \mathbb{R}^+ , is a subset of the codomain which is \mathbb{R} .

Up to now we have discussed functions given by a single formula such as those in Example 7. However there are many functions which have different formulae for different values of x (the domain). For example the modulus function is defined by

$f : \mathbb{R} \rightarrow \mathbb{R}$ given as

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

What does this mean?

It means that if x is positive or zero then $f(x) = x$ and if x is negative then $f(x) = -x$. We examine this modulus function later in this chapter.

In the next example we discuss Dirichlet's function named after the Belgium mathematician Dirichlet.



Fig 10 Dirichlet
from 1805 to 1859

Johann Dirichlet was born in Düren, now in Germany, and by the age of 12 he had developed a love for mathematics spending his pocket money on buying mathematics books. After his school years he went to study mathematics in Paris at Collège de France and Faculté des Sciences where he made contact with the well established mathematicians of that era such as Fourier, Laplace, Legendre and Poisson.

In 1828 he became professor of mathematics at the University of Berlin and remained there for 27 years. In 1855 he moved to the University of Göttingen, Germany where he had more time to do research. Dirichlet worked in many fields of mathematics – number theory, analysis, partial differential equations etc but he is particularly remembered for his contribution to analytic number theory.

Example 8

Dirichlet's Function is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Determine $f(2)$ and $f(\sqrt{2})$. Also find the range of the function.

Solution

What does the Dirichlet's function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

mean?

It says that if x is a rational number then $f(x) = 1$ and if x is an irrational number then $f(x) = 0$. *What is the value of $f(x)$ if $x = 2$?*

$f(2) = 1$ because 2 is a rational number.

If $x = \sqrt{2}$ then $f(\sqrt{2}) = 0$ because $\sqrt{2}$ is an irrational number.

What values can

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

take for any real x ?

Clearly the function f can only have two values 0 and 1. Therefore the range of f is the set $\{0, 1\}$.

SUMMARY

A function $f : A \rightarrow B$ is a rule which assigns every element in set A to a unique element in set B. Set A is called the **domain** of the function and set B is called the **codomain** of the function.

The range of a function f is the set of all the values $f(x)$.