Chapter 2 : Infinite Series

Section C Properties of Convergent and Divergent Series

By the end of this section you will be able to

- prove certain properties of convergent series
- apply divergent tests to a given series
- use sum and scalar product rules for testing convergence and determine the sum of a series

C1 Convergent Series

In this subsection we prove certain properties of convergent series. To prove these we need to apply the following results about limits of sequences from chapter 1.

If sequences (S_n) and (T_n) converge then

(1.??) $$\lim_{n\to\infty}(S_n \pm T_n) = \lim_{n\to\infty}(S_n) \pm \lim_{n\to\infty}(T_n)$$

(1.??) $$\lim_{n\to\infty}(cS_n) = c\lim_{n\to\infty}(S_n) \text{ where } c \text{ is a constant}$$

We will be applying these results to sequences of partial sums, (S_n), throughout this section. Remember (S_n), the nth partial sum, is a sequence.

Of course to prove any results about series we need to consider the general series such as $\sum_{k=1}^{\infty}(a_k)$ rather than a particular series. For clarity of understanding we will sometimes use the following convenient notation:

$$\sum_{k=1}^{\infty}(a_k) = \sum(a_k) = \sum a$$

It is automatically understood that both the series on the Right, $\sum(a_k)$ and $\sum a$, are identical to the infinite series on the Left. The reason for this is that many students are lost in notation and see all these symbols and are “put off” the topic of infinite series for good. Don’t lose yourself in all this notation, the mathematics in this section is generally straightforward.

Proposition (2.5). If the series $\sum_{k=1}^{\infty}(a_k)$ is convergent then

$$\lim_{n\to\infty}(a_n) = 0$$

Note: *What does this proposition mean?*

It says that if the infinite series $\sum_{k=1}^{\infty}(a_k)$ converges then the terms eventually get smaller and smaller in size and tend to zero for large k. This proposition (2.5) is called the **nth term test**.

Proof. Let S_n be the nth partial sum, that is the sum of the first n terms:

$$S_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n}(a_k)$$

Since we are told that $\sum_{k=1}^{\infty}(a_k)$ converges therefore $\lim_{n\to\infty}(S_n)$ exists because

$$\lim_{n\to\infty}(S_n)=\sum_{k=1}^{\infty}(a_k)$$

Let L be this limit and so

$$\lim_{n\to\infty}(S_n)=L \qquad (*)$$

As $n\to\infty$ then $n-1\to\infty$ therefore

$$\lim_{n\to\infty}(S_{n-1})=L \qquad (**)$$

as well. But we need to show

$$\lim_{n\to\infty}(a_n)=0$$

How can we show this?

We need to consider (a_n) rather than (S_n). *How can we write* (a_n) *?*

We want to express a_n in terms of S_n and remember S_n is the sum of the first n terms. We write a_n as the sum of the first n terms **minus** the sum of the first $n-1$ terms, that is:

$$\begin{aligned} a_n &= \underbrace{(a_1+a_2+a_3+\ldots+a_n)}_{=S_n}-\underbrace{(a_1+a_2+a_3+\ldots+a_{n-1})}_{=S_{n-1}} \\ &= S_n - S_{n-1} \qquad [\text{Expressing the Sum in Terms of } S_n\text{'s}] \end{aligned}$$

Since we are interested in the limiting value of a_n we have

$$\begin{aligned} \lim_{n\to\infty}(a_n) &= \lim_{n\to\infty}(S_n-S_{n-1}) \\ &= \lim_{n\to\infty}(S_n)-\lim_{n\to\infty}(S_{n-1}) \qquad [\text{By Limits of Sequences Result (1.??)}] \\ &= L-L \qquad [\text{By } (*) \text{ and } (**)] \\ &= 0 \end{aligned}$$

Hence we have the required result, that is if $\sum_{k=1}^{\infty}(a_k)$ converges then $\lim_{n\to\infty}(a_n)=0$.

■

However the condition

$$\lim_{n\to\infty}(a_n)=0$$

is NOT sufficient for the convergent of the series $\sum_{k=1}^{\infty}(a_k)$. That is the converse of proposition (2.5) is NOT true. For example if we consider the harmonic series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)$$

then $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$ but the series $\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)$ diverges. (See Example 9).

(1.??) $\lim_{n\to\infty}(S_n\pm T_n)=\lim_{n\to\infty}(S_n)\pm\lim_{n\to\infty}(T_n)$

Hence if the series $\sum_{k=1}^{\infty}(a_k)$ is convergent then

$$\lim_{n\to\infty}(a_n)=0 \quad \text{[Terms tend to 0]}$$

However if $\lim_{n\to\infty}(a_n)=0$ then the series $\sum_{k=1}^{\infty}(a_k)$ may NOT converge.

This is critical because many students wrongly think that if the nth terms eventually go to zero then the series converges. Proposition (2.5) says that if the series converges then the nth term eventually goes to zero. But if the nth term goes to zero as $n\to\infty$ that does **not** mean the series converges. Another example is $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}\right)$. We know

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)=0 \quad \text{[Terms eventually go to Zero]}$$

but the series $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}\right)$ is divergent. You were asked to show the divergence of this series, $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}\right)$, in question 6 of Exercise 2b.

Note that in the above we can replace the letter k by n without changing the meaning. It does **not** make any difference. The letter k is a 'dummy variable' and can be replaced by any symbol. For example proposition (2.5) is the following:

If $\sum_{n=1}^{\infty}(a_n)$ converges then $\lim_{n\to\infty}(a_n)=0$.

The letter k was used because n represented the nth partial sum, S_n.

C2 **Divergent Tests**

Proposition (2.6). If $\lim_{n\to\infty}(a_n)\neq 0$ then $\sum_{n=1}^{\infty}(a_n)$ diverges.

Proof. This proposition is the contrapositive of proposition (2.5).

■

What does the term contrapositive mean?

We discussed contrapositive in the chapter on logic. Let A and B be statements then the contrapositive of

'If A then B' is 'If (not B) then (not A)'. These are logically equivalent.

In proposition (2.5), A is the statement '$\sum_{n=1}^{\infty}(a_n)$ converges' and B is the statement '$\lim_{n\to\infty}(a_n)=0$'. *What is 'not B' in this case?*

$$\lim_{n\to\infty}(a_n)\neq 0$$

What is 'not A' in this case?

$$\sum_{n=1}^{\infty}(a_n) \text{ diverges}$$

Hence the contrapositive of proposition (2.5) is proposition (2.6).

(2.5) If $\sum_{n=1}^{\infty}(a_n)$ converges then $\lim_{n\to\infty}(a_n)=0$

Proposition (2.7). If $\lim_{n\to\infty}(a_n)$ does not exist then $\sum_{n=1}^{\infty}(a_n)$ diverges.

Proof. Suppose $\sum_{n=1}^{\infty}(a_n)$ converges then by proposition (2.5) we have

$$\lim_{n\to\infty}(a_n)=0$$

This contradicts that the $\lim_{n\to\infty}(a_n)$ does not exist. Thus proposition (2.7) is established.

■

Note that if $\lim_{n\to\infty}(a_n)$ is **not** zero or does **not** exist then we can conclude the series $\sum_{n=1}^{\infty}(a_n)$ diverges.

We use propositions (2.6) and (2.7) as divergence tests for a given series.

Example 10.

Show that the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)$ diverges.

Solution. Since

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) \quad \begin{bmatrix}\text{Dividing Numerator}\\ \text{and Denominator by } n\end{bmatrix}$$

$$=\left(\frac{1}{1+0}\right)=1\neq 0$$

Therefore by proposition (2.6) the given series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)$ diverges.

Example 11.

Show that the series $\sum_{n=1}^{\infty}\left(\frac{1-n^2}{n^2+5n}\right)$ diverges.

Solution. We have

$$\lim_{n\to\infty}\left(\frac{1-n^2}{n^2+5n}\right)=\lim_{n\to\infty}\left(\frac{1/n^2-1}{1+5/n}\right) \quad \begin{bmatrix}\text{Dividing Numerator}\\ \text{and Denominator by } n^2\end{bmatrix}$$

$$=\frac{0-1}{1+0}=-1\neq 0$$

By proposition (2.6) the given series $\sum_{n=1}^{\infty}\left(\frac{1-n^2}{n^2+5n}\right)$ diverges.

(2.5) If $\sum_{n=1}^{\infty}(a_n)$ converges then $\lim_{n\to\infty}(a_n)=0$

(2.6) If $\lim_{n\to\infty}(a_n)\neq 0$ then $\sum_{n=1}^{\infty}(a_n)$ diverges.

Example 12.

Show that the series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Solution.

Since $\lim_{n\to\infty}(-1)^n$ does **not exist**, therefore by proposition (2.7) the given series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

C3 **Rules of Convergent Series**

Proposition (2.8). Sum Rule. If both series $\sum_{k=1}^{\infty}(a_k)$ and $\sum_{k=1}^{\infty}(b_k)$ are convergent then $\sum_{k=1}^{\infty}(a_k+b_k)$ is convergent and

$$\sum_{k=1}^{\infty}(a_k+b_k)=\sum_{k=1}^{\infty}(a_k)+\sum_{k=1}^{\infty}(b_k)$$

Proof. Let $S_n=\sum_{k=1}^{n}(a_k)$ and $T_n=\sum_{k=1}^{n}(b_k)$. Both sequences $\lim_{n\to\infty}(S_n)$ and $\lim_{n\to\infty}(T_n)$ are convergent because both series converge. Consider

$$\begin{aligned}\sum_{k=1}^{\infty}(a_k+b_k)&=\lim_{n\to\infty}\left[\sum_{k=1}^{n}(a_k+b_k)\right]\\ &=\lim_{n\to\infty}\left[\sum_{k=1}^{n}(a_k)+\sum_{k=1}^{n}(b_k)\right] && \left[\text{Taking in }\sum\text{ because Finite Sums}\right]\\ &=\lim_{n\to\infty}\left[S_n+T_n\right] && \left[\text{From Above}\right]\\ &=\lim_{n\to\infty}(S_n)+\lim_{n\to\infty}(T_n) && \left[\text{Applying the Limits of Sequence (1.??)}\right]\\ &=\sum_{k=1}^{\infty}(a_k)+\sum_{k=1}^{\infty}(b_k) && \left[\text{Because }\lim_{n\to\infty}(S_n)=\sum_{k=1}^{\infty}(a_k)\text{ and similarly for }b_k\right]\end{aligned}$$

Hence we have our required result.

■

Proposition (2.8) is called the sum rule and is used to test the convergence of a given series. *What does proposition (2.8) mean in everyday language?*

It says you can split up a series such as $\sum(a+b)$ into $\sum a$ + $\sum b$ if they both converge. This is very useful in a series with many parts because we can examine just one part at a time rather than the whole series in one go. It is more digestible if you consider each part separately.

In proposition (2.8) the plus sign in the middle can be replaced by a minus sign, that is

---

(1.??) $\lim_{n\to\infty}(S_n\pm T_n)=\lim_{n\to\infty}(S_n)\pm\lim_{n\to\infty}(T_n)$

(2.7) If $\lim_{n\to\infty}(a_n)$ does **not** exist then $\sum_{n=1}^{\infty}(a_n)$ diverges.

$$\sum (a-b)=\sum a-\sum b$$

You are asked to prove this in Exercise 2c.

Proposition (2.9). Scalar Product Rule. If $\sum_{k=1}^{\infty}(a_k)$ converges then

$$\sum_{k=1}^{\infty} c(a_k)=c\sum_{k=1}^{\infty}(a_k) \text{ where } c \text{ is a constant}$$

Proof. See Exercise 2c.

Proposition (2.9) is called the scalar product rule. We can also apply this scalar product rule to test a given series for convergence. *What does the scalar product rule mean?*

If a series $\sum c(a)$, where c is a constant, converges then we can take the constant c outside the summation sign, $\sum$, that is

$$\sum c(a)=c\sum a$$

Normally we use **both** these, sum and scalar product, rules together to breakdown the given series and then analyse each series separately. In the following examples we apply these rules.

Example 13

Test

$$\sum_{k=1}^{\infty}\left[\left(\frac{2}{k(k+1)}\right)+\left(\frac{1}{2}\right)^k\right]$$

for convergence. If the series converges determine its sum.

Solution.

From Example 7 we know

$$\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right)=1$$

This is 2 times the first part of the given series. Because $\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right)=1$ converges therefore we can apply the scalar product rule

(2.9) $$\sum c(a)=c\sum a$$

Hence in the first part of the series we can take the constant 2 out and we have

$$\sum_{k=1}^{\infty}\left(\frac{2}{k(k+1)}\right)=2\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right)=2(1)=2 \qquad (*)$$

What do you notice about the remaining, $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ *?*

Well $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ is a geometric series with a common ratio $r=\frac{1}{2}$ and first term $a=\frac{1}{2}$.
Since the common ratio $|r|$ is less than 1 therefore the series converges and the sum is given by

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1}=\frac{a}{1-r} \qquad (|r|<1)$$

We have

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k=\frac{1/2}{1-1/2}=1 \qquad (**)$$

Since both $\sum_{k=1}^{\infty}\left(\frac{2}{k(k+1)}\right)$ and $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ converge we can apply the sum rule

(2.8) $$\sum(a+b)=\sum(a)+\sum(b)$$

to the given series:

$$\sum_{k=1}^{\infty}\left[\left(\frac{2}{k(k+1)}\right)+\left(\frac{1}{2}\right)^k\right]=\sum_{k=1}^{\infty}\left(\frac{2}{k(k+1)}\right)+\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$$

$$=2+1=3 \quad [\text{By (*) and (**)}]$$

Notice how we split the series up into parts in Example 13. If the series in each part converges then we can take out a constant and breakdown the addition (or subtraction).
We can combine the sum and scalar product rules, that is propositions (2.8) and (2.9), to give the following result:

Proposition (2.10). If both $\sum a_k$ and $\sum b_k$ converge then

$$\sum\left[c(a_k)+d(b_k)\right]=c\sum(a_k)+d\sum(b_k)$$

where c and d are constants.
Also true for subtraction, that is

$$\sum\left[c(a_k)-d(b_k)\right]=c\sum(a_k)-d\sum(b_k)$$

Proof. See Exercise 2c.

What does proposition (2.10) mean?
It says you can take the $\sum$ inside the addition (or subtraction) and take the constants out provided the individual series converge. This is analogous to integration where you can carry out the integration under the addition and take out the constants.
Generally it is easier to use (2.10) directly as the next example shows.

Example 14
Test the following series for convergence

$$\sum_{n=1}^{\infty}\left(\frac{5^{n-3}+7^{n-1}}{8^n}\right)$$

If the series converges determine its sum.
Solution.
Do we need to apply the sum and scalar product rules to test the given series for convergence?
Yes because we can breakdown the general term as:

$$\begin{aligned}\frac{5^{n-3}+7^{n-1}}{8^n}&=\frac{5^{n-3}}{8^n}+\frac{7^{n-1}}{8^n}\\&=5^{-3}\left(\frac{5^n}{8^n}\right)+7^{-1}\left(\frac{7^n}{8^n}\right)\qquad\begin{bmatrix}\text{Using the Rules of Indices}\\ a^m a^n=a^{m+n}\end{bmatrix}\\&=\frac{1}{125}\left(\frac{5}{8}\right)^n+\frac{1}{7}\left(\frac{7}{8}\right)^n\qquad(\dagger)\end{aligned}$$

The last line follows by using the rules of indices, $5^{-3}=\frac{1}{5^3}=\frac{1}{125}$ and $7^{-1}=\frac{1}{7^1}=\frac{1}{7}$.

Both $\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n$ and $\sum_{n=1}^{\infty}\left(\frac{7}{8}\right)^n$ are geometric series with a common ratio $r=\frac{5}{8}$ and $r=\frac{7}{8}$ respectively. Since the common ratio $|r|<1$ in both cases, these series converge and their sum is determined by applying

(2.3) $$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}\qquad(|r|<1)$$

as follows:

$$\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n=\frac{5/8}{1-5/8}=\frac{5}{3}$$

$$\sum_{n=1}^{\infty}\left(\frac{7}{8}\right)^n=\frac{7/8}{1-7/8}=7$$

We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{5^{n-3}+7^{n-1}}{8^n}\right)&=\sum_{n=1}^{\infty}\left[\frac{1}{125}\left(\frac{5}{8}\right)^n+\frac{1}{7}\left(\frac{7}{8}\right)^n\right]\qquad[\text{By }(\dagger)]\\&=\frac{1}{125}\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n+\frac{1}{7}\sum_{n=1}^{\infty}\left(\frac{7}{8}\right)^n\qquad[\text{By (2.10)}]\\&=\frac{1}{125}\left(\frac{5}{3}\right)+\frac{1}{7}(7)=\frac{76}{75}\end{aligned}$$

The last line follows from above because $\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n=\frac{5}{3}$ and $\sum_{n=1}^{\infty}\left(\frac{7}{8}\right)^n=7$. Hence the given series, $\sum_{n=1}^{\infty}\left(\frac{5^{n-3}+7^{n-1}}{8^n}\right)$, converges with sum $\frac{76}{75}$.

SUMMARY

If a series $\sum a_k$ is convergent then

$$\lim_{n\to\infty}(a_n)=0$$

But the converse of this is not true. That is if $\lim_{n\to\infty}(a_n)=0$ then $\sum(a_k)$ may NOT converge.

But if $\lim_{n\to\infty}(a_n)\neq 0$ or it does not exist then $\sum(a_n)$ diverges.

It is important to note that if $\lim_{n\to\infty}(a_n)=0$ then we **cannot** conclude whether the series $\sum(a_n)$ converges or diverges. But if $\lim_{n\to\infty}(a_n)\neq 0$, or does not exist, then we can say $\sum(a_n)$ diverges.

If $\sum(a_k)$ and $\sum(b_k)$ are both convergent then $\sum(a_k+b_k)$ is convergent and

(2.8) $$\sum(a_k+b_k)=\sum(a_k)+\sum(b_k)$$ [Sum Rule]

(2.9) $$\sum c(a_k)=c\sum(a_k)$$ [Scalar Product Rule]

where c is a constant.

(2.10) $$\sum\left[c(a_k)\pm d(b_k)\right]=c\sum(a_k)\pm d\sum(b_k)$$ where c and d are constants