Solutions to Exercise 2c

1\. (a) $\lim_{n\to\infty}\left(\frac{2n-1}{n}\right) = \lim_{n\to\infty}\left(2-\frac{1}{n}\right) = 2$. This means the nth term does not go to zero therefore $\sum\left(\frac{2n-1}{n}\right)$ diverges.

(b) Similarly $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$ so $\sum\left(1+\frac{1}{n}\right)$ diverges.

(c) Also

$$\lim_{n\to\infty}\left(\frac{2n^2-n+1}{n^2+n+1}\right) = \lim_{n\to\infty}\left(\frac{2-1/n+1/n^2}{1+\frac{1}{n}+1/n^2}\right)$$

$$= 2$$

The given series diverges.

(d) We are given $\sum_{n=1}^{\infty}\left(\frac{3n+1}{5n-1}\right)$. The limit of the sequence

$$\lim_{n\to\infty}\left(\frac{3n+1}{5n-1}\right) = \lim_{n\to\infty}\left(\frac{3+1/n}{5-\frac{1}{n}}\right) = \frac{3}{5}$$

Series diverges.

(e) Clearly $\lim_{n\to\infty}(2)^n \neq 0$. Hence we have divergence.

(f) $\lim_{n\to\infty}\left(\frac{7}{6}\right)^n \neq 0$ therefore $\sum_{n=1}^{\infty}\left(\frac{7}{6}\right)^n$ diverges.

(g) $\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right) = \lim_{n\to\infty}\left(\frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}\right) = 1 \neq 0. \Rightarrow$ divergence.

(h) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$ therefore $\sum 1+\frac{1}{n}$ diverges.

(i) $\lim_{n\to\infty}\cos(n\pi)$ does not exist therefore $\sum\cos(n\pi)$ diverges.

(j) Similarly $\sum\sin(n\pi)$ diverges.

2. If we can show that $\lim_{n\to\infty} \sqrt{\frac{n-1}{n+1}} \neq 0$ then we ~~can~~ conclude $\sum \sqrt{\frac{n-1}{n+1}}$ diverges. We have

$$\lim_{n\to\infty} \sqrt{\frac{n-1}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n-1}{n+1}} \cdot \sqrt{\frac{n+1}{n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{(n-1)(n+1)}{(n+1)^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2\left(1-\frac{1}{n^2}\right)}}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \left(1-\frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)\left(1-\frac{1}{n^2}\right)$$

$$= (1)(1) = 1$$

Therefore $\sum \sqrt{\frac{n-1}{n+1}}$ diverges.

3. (a) Let S_n and T_n be the partial sums

$$S_n = \sum_{k=1}^{n} a_k \text{ and } T_n = \sum_{k=1}^{n} b_k$$

Since these are partial sums we have

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k - b_k)$$

Taking the limit:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}(a_k-b_k)\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k\right)$$

Since $\sum a_k$ and $\sum b_k$ are convergent we can take the limit in:

$$=\lim_{n\to\infty}\left(\sum_{k=1}^{n}a_k\right)-\lim_{n\to\infty}\left(\sum_{k=1}^{n}b_k\right)$$

$$\sum_{k=1}^{\infty}(a_k-b_k)=\sum_{k=1}^{\infty}a_k-\sum_{k=1}^{\infty}b_k$$

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(b) Similarly

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}(ca_k)\right)=c\lim_{n\to\infty}\left(\sum_{k=1}^{n}a_k\right)$$

(c) Applying parts (a) & (b) gives (c).

4. Error occurs in the second line. You can see this if you expand.

$$\sum_{k=1}^{n}(a_k b_k)=a_1b_1+a_2b_2+a_3b_3+\cdots+a_nb_n$$

$$\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k=(a_1+a_2+a_3+\cdots+a_n)(b_1+b_2+b_3+\cdots+b_n)$$

$$\neq a_1b_1+a_2b_2+a_3b_3+\cdots+a_nb_n$$

5. Since $\sum e^{-n}$ and $\sum \pi^{-n}$ are geometric series & common ratio $r<1$ we have

$$\sum e^{-n}=\sum\left(\frac{1}{e^n}\right)=\frac{1/e}{1-\frac{1}{e}}=\frac{1}{e-1}$$

$$\sum \pi^{-n}=\sum\frac{1}{\pi^n}=\frac{1/\pi}{1-\frac{1}{\pi}}=\frac{1}{\pi-1}$$

Both these series converge therefore

$$\sum \left(e^{-n} + \pi^{-n}\right) = \sum e^{-n} + \sum \pi^{-n}$$

$$= \frac{1}{e-1} + \frac{1}{\pi-1} = \frac{(\pi-1)+(e-1)}{(\pi-1)(e-1)}$$

$$= \frac{\pi + e - 2}{\pi e - \pi - e + 1}$$

(b) By writing $\frac{1}{4n^2-1}$ in terms of partial fractions:

$$\frac{1}{4n^2-1} = \frac{\cancel{\$}\,1}{(2n-1)(2n+1)}$$

$$= \frac{A}{2n-1} + \frac{B}{2n+1}$$

Cover up gives $A = \frac{1}{2}$, $B = -\frac{1}{2}$

$$\frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

Consider the partial sum

$$\sum_{n=1}^{m} \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] = \frac{1}{2}\Bigg[\underbrace{\left(1-\frac{1}{3}\right)}_{n=1} + \underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{n=2}$$

$$+ \underbrace{\left(\frac{1}{5}-\frac{1}{7}\right)}_{n=3} + \dots + \underbrace{\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right)}_{n-1}$$

$$+ \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\Bigg]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{2}\left[\frac{1}{2n-2} - \frac{1}{2n+1}\right] = \frac{1}{2}\left[1 - \lim_{n\to\infty}\left(\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}$$

$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is a geometric series with $a=\frac{2}{3}$ & $r=\frac{2}{3}<1$

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2/3}{1-2/3} = \frac{2}{3-2} = \frac{2}{1\cancel{3}} = 2$$

Hence

$$\sum_{n=1}^{\infty}\left[\frac{1}{4n^2-1}+\left(\frac{2}{3}\right)^n\right] = \frac{1}{2}+\cancel{\frac{2}{3}}2 = 2\tfrac{1}{2} \text{ or } \frac{5}{2}$$

(c) We rewrite

$$\frac{4^{n-2}+6^{n-1}}{12^n} = \frac{4^{n-2}}{12^n}+\frac{6^{n-1}}{12^n}$$

$$= 4^{-2}\left(\frac{4}{12}\right)^n + 6^{-1}\left(\frac{6}{12}\right)^n$$

$$= \frac{1}{16}\left(\frac{4}{12}\right)^n + \frac{1}{6}\left(\frac{6}{12}\right)^n$$

$$= \frac{1}{16}\left(\frac{1}{3}\right)^n + \frac{1}{6}\left(\frac{1}{2}\right)^n$$

$\sum\left(\frac{1}{3}\right)^n$ and $\sum\left(\frac{1}{2}\right)^n$ are both geometric series therefore

$$\sum\left(\frac{1}{3}\right)^n = \frac{1/3}{1-1/3} \quad \left[\text{Using } S_\infty = \frac{a}{1-r}\right]$$

$$= \frac{1}{3-1} = \frac{1}{2}$$

Also

$$\sum\left(\frac{1}{2}\right)^n = \frac{1/2}{1-1/2} = \frac{1}{2-1} = 1$$

Using these results on the given series:

$$\sum_{n=1}^{\infty}\frac{4^{n-2}+6^{n-1}}{12^n} = \frac{1}{16}\left(\frac{1}{2}\right)+\frac{1}{6}(1)$$

$$= \frac{1}{32}+\frac{1}{6} = \frac{19}{96}$$

(e) Using partial fractions on the first term:

$$\frac{4}{3n(n+2)} = \frac{A}{3n} + \frac{B}{n+2}$$

Cover up yields $A = 2, \quad B = -\frac{4}{6} = -\frac{2}{3}$

$$\frac{4}{3n(n+2)} = 2\left[\frac{1}{3n} - \frac{1}{3(n+2)}\right]$$

Consider the partial sum:

$$\sum_{k=1}^{n} \frac{1}{3k(k+2)} = 2\Bigg[\underbrace{\left(\frac{1}{3} - \frac{1}{3(3)}\right)}_{k=1} + \underbrace{\left(\frac{1}{6} - \frac{1}{12}\right)}_{k=2}$$

$$+ \underbrace{\left(\frac{1}{9} - \frac{1}{15}\right)}_{k=3} + \underbrace{\left(\frac{1}{12} - \frac{1}{18}\right)}_{k=4} + \cdots$$

$$+ \underbrace{\left(\frac{1}{3n-3} - \frac{1}{3n+1}\right)}_{k=n-1} + \left(\frac{1}{3n} - \frac{1}{3n+6}\right)\Bigg]$$

$$= 2\left[\frac{1}{3} + \frac{1}{6} - \frac{1}{3n+1} + \frac{1}{3n} - \frac{1}{3n+6}\right]$$

The infinite sum is given by

$$\sum_{k=1}^{\infty} \frac{1}{3k(k+2)} = \lim_{n\to\infty} 2\left[\frac{1}{3} + \frac{1}{6} - \frac{1}{3n+1} + \frac{1}{3n} - \frac{1}{3n+6}\right]$$

$$= 2\left(\frac{1}{3} + \frac{1}{6}\right) = 1$$

Considering the other term we have

$$\sum_{n=1}^{\infty} 2\left(\pi^{-2n}\right) = \sum_{n=1}^{\infty} \frac{2}{\pi^{2n}} = \sum_{n=1}^{\infty} \frac{2}{(\pi^2)^n}$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{1}{\pi^2}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\pi^2}\right)^n = \frac{1/\pi^2}{1-\frac{1}{\pi^2}} = \frac{1}{\pi^2-1}$$

Hence

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{\pi^2}\right)^n = \frac{2}{\pi^2-1}.$$

The sum of the combined series is

$$\sum_{n=1}^{\infty} \left(\frac{4}{3n(n+2)} + 2\pi^{-2n}\right) = 1 + \frac{2}{\pi^2-1}$$

$$= \frac{\pi^2-1+2}{\pi^2-1} = \frac{\pi^2+1}{\pi^2-1}$$

6. (a) Note $\cos^2(x) + \sin^2(x) = 1$ therefore we have

$$\sum_{n=1}^{\infty} \left(\cos^2(x) + \sin^2(x)\right)^n = \sum 1^n$$

Since $\lim_{n\to\infty} 1^n \neq 0$ so the given series diverges.

(b) From complex nos we have

$$e^{i\pi} = -1$$

$$\sum_{n=1}^{\infty} (e^{i\pi})^n = \sum_{n=1}^{\infty} (-1)^n$$

Similarly $\lim_{n\to\infty} (-1)^n \neq 0$. This means $\sum (e^{i\pi})^n$ diverges.

7. Consider the infinite series

$$\sum_{n=1}^{\infty} (x^n) = x + x^2 + x^3 + \cdots$$

This is a GS with common ratio $r = x$. For $|r| = |x| < 1$ the GS converges. If a series converges then it's nth term tends to zero. Hence

$$\lim_{n\to\infty} (x^n) = 0$$

8. Using the hint given in the question we have

$$n = 1 + x \quad \text{where } x \geq 0$$

Applying binomial we have

$$n^{1/n} = (1+x)^{1/n}$$

$$= 1 + \frac{1}{n}x + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}x^2 + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{3!}x^3 + \cdots$$

Applying limits:

$$\lim_{n\to\infty}\left(n^{1/n}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n}x + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!} + \cdots\right)$$

$$= 1$$

Hence $\sum n^{1/n}$ diverges.

9. A well known limit is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

Hence $\sum \left(1+\frac{1}{n}\right)^n$ diverges.