Complete Solutions to Exercise 2d

1. We use the $p$-test in each case:

\[ \sum \frac{1}{n^p} \begin{cases} 
\text{converges if } p > 1 \\
\text{diverges if } p \leq 1
\end{cases} \]

(a) Since $p = 3 > 1$, $\sum \frac{1}{n^3}$ converges.

(b) Similarly $p = 4$, $\sum \frac{1}{n^4}$ ---

(c) We have $p = \frac{1}{3} < 1$ therefore $\sum \frac{1}{n^{1/3}}$ diverges.

(d) We have $p = e > 1$ --- $\sum \frac{1}{n^e}$ converges.

(e) " " $p = \pi > 1$ " " $\sum \frac{1}{n^\pi}$ " "

(f) Fundamental trigonometric identity

\[ \cos^2(x) + \sin^2(x) = 1 \]

We have \[ \sum \frac{1}{n \cos^2(x) + \sin^2(x)} = \sum \frac{1}{n} = \sum \frac{1}{n}. \] This is the harmonic series and diverges.

(g) Using the rules of indices we have

\[ \sqrt[3]{n} \sqrt[n]{n} = n^{\frac{1}{3} + \frac{1}{n}} = n^{\frac{1}{3} + \frac{1}{n}} = n^{\frac{5}{6}} \]

In this case $p = \frac{5}{6} < 1$ therefore $\sum \frac{1}{n^{5/6}}$ diverges.

(h) Again applying the rules of indices:

\[ \frac{\sqrt[n]{n^3}}{\sqrt[n]{n}} = \frac{n^{\frac{1}{2}}} {n^{\frac{1}{2}}} = n^{\frac{1}{2} - \frac{1}{2}} = n^{-1/6} \]

\[ \sum \frac{\sqrt[n]{n^3}}{\sqrt[n]{n}} = \sum n^{-1/6} = \sum \frac{1}{n^{1/6}} \text{ diverges because } p = \frac{1}{6} < 1. \]
(i) Using indices we have

\[ \frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}} \]

\[ \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}} \text{ converges because } p = \frac{3}{2} > 1. \]

(ii) \( \frac{1}{n^{5/4}} = \frac{1}{n^{1.25}} \). For \( n \geq 2 \) we have

\[ \frac{1}{n^{5/4}} = \]

& \( n^{1/2} > 1 \) so \( \sum \frac{1}{n^{5/4}} \) converges. Hence

\[ \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} = \frac{1}{1^{5/4}} + \frac{1}{2^{5/4}} + \frac{1}{3^{5/4}} + \cdots \]

\[ = 1 + \frac{1}{2^{5/4}} + \frac{1}{3^{5/4}} + \cdots \]

\[ = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{5/4}} \]

Note \( n^{1/2} > 1 \) so \( \sum_{n=2}^{\infty} \frac{1}{n^{1/2}} \) converges. Hence \( \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \) converges.
2. Use the comparison test in each case.

If \( 0 \leq a_n \leq b_n \) then

(i) \( \sum b_n \) converges \( \Rightarrow \) \( \sum a_n \) converges

(ii) \( \sum a_n \) diverges \( \Rightarrow \) \( \sum b_n \) diverges

(a) Since \( n \in \mathbb{N} \) we have

\[
\frac{1}{n^2+n} < \frac{1}{n^2}
\]

By the \( p \) test we know \( \sum \frac{1}{n^2} \) converges. Therefore by the comparison test \( \sum \frac{1}{n^2+n} \) converges.

(b) We have the inequality

\[
3^n+n > 3^n
\]

\[
\frac{1}{3^n+n} < \frac{1}{3^n}
\]

The geometric series \( \sum \frac{1}{3^n} \) converges because \( r = \frac{1}{3} < 1 \). By the comparison test

\[
\sum \frac{1}{3^n+n} \]

converges.

(c) We can rewrite

\[
\frac{1+4^n}{3^n} = \frac{1}{3^n} + \left(\frac{4}{3}\right)^n > \left(\frac{4}{3}\right)^n
\]

\( \sum \left(\frac{4}{3}\right)^n \) is a geometric series with common ratio \( r = \frac{4}{3} > 1 \). Hence \( \sum \left(\frac{4}{3}\right)^n \) diverges. By the comparison test

\[
\sum \frac{1+4^n}{3^n}
\]

diverges.
(d) We have the inequality
\[ 1 + 4^n > 4^n \]
\[ \frac{1}{1 + 4^n} < \frac{1}{4^n} \]
The series \( \sum \frac{1}{4^n} = \sum \left( \frac{1}{4} \right)^n \) converges because it is a geometric series with \( r = \frac{1}{4} \). By the comparison test
\[ \sum \frac{1}{1 + 4^n} \] converges.

(e) We have
\[ 3n-1 < 3n \]
\[ \frac{1}{3n-1} > \frac{1}{3n} \]
The series \( \sum \frac{1}{3n} \) diverges. By the comparison test
\[ \sum \frac{1}{3n-1} \] diverges.

(f) We have the inequality
\[ n! \cdot n > n^2 \]
\[ \frac{1}{n! \cdot n} \leq \frac{1}{n^2} \]
\[ \sum \frac{1}{n^2} \] converges by the p-test because \( p=2>1 \). Therefore by the comparison test
\[ \sum \frac{1}{n! \cdot n} \] converges.

(g) We have
\[ n-1 < n \]
\[ \sqrt[n-1]{n^n} \]
\[ \frac{1}{2^{n-1}} > \frac{1}{2^n} \]
\[ \sum \frac{1}{n^{3/2}} = \sum \frac{1}{n^{3/2}} \text{ diverges because } p = \frac{3}{2} < 1. \]

By the comparison test
\[ \sum \frac{1}{\sqrt{n}} \text{ diverges.} \]

(h) We have
\[ 5^n n! > 5^n \]
\[ \frac{1}{5^n n!} < \frac{1}{5^n} \]

The series \( \sum \frac{1}{5^n} \) converges because it is a geometric series with common ratio \( r = \frac{1}{5} < 1 \).

By the comparison test
\[ \sum \frac{1}{5^n n!} \text{ converges.} \]

(i) Similarly
\[ 2^n + 2n > 2^n \]
\[ \frac{1}{2^n + 2n} < \frac{1}{2^n} \]

Since \( \sum \frac{1}{2^n} \) converges (as with \( r = \frac{1}{2} < 1 \)) so
\[ \sum \frac{1}{2^n + 2n} \text{ converges.} \]

(j) Examining the denominator term
\[ n^{3+1} > n^5 \]
\[ \frac{n^3 + 1}{n^{5+1}} < \frac{n^3 + 1}{n^5} = \frac{1}{n^2} + \frac{1}{n^5} \]

By the p test \( \sum \frac{1}{n^4} \) and \( \sum \frac{1}{n^5} \) converge therefore
\[ \sum \frac{1}{n^4 + \frac{1}{n^5}} \text{ converges.} \]

By the comparison test
\[ \sum \frac{n^3 + 1}{n^{5+1}} \text{ converges.} \]
(k) We have the inequality
\[ n < 2^n \quad \text{where } n \in \mathbb{N} \]
Therefore
\[ \frac{n + 2^n}{3^n} < \frac{2^n + 2^n}{3^n} = \frac{2 \cdot 2^n}{3^n} = 2 \left( \frac{2}{3} \right)^n \]

The infinite series \( \sum 2 \left( \frac{2}{3} \right)^n \) converges because it is a G.S. with common ratio \( r = \frac{2}{3} < 1 \). By the comparison test
\[ \sum \frac{n + 2^n}{3^n} \text{ converges.} \]

3. In these cases we can use the limit comparison test:

Let \( a_n \geq 0 \) and \( b_n > 0 \) and
\[ \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = L \]
(i) If \( L > 0 \) then \( \sum a_n \) is convergent \( \iff \sum b_n \) is convergent.
(ii) If \( L = 0 \) then \( \sum a_n \) is divergent \( \iff \sum b_n \) is divergent.

(a) Let \( a_n = \frac{1}{n^3} \) and \( b_n = \frac{1}{n^3 - n} \). Then
\[ \lim_{n \to \infty} \frac{\frac{1}{n^3}}{\frac{1}{n^3 - n}} = \lim_{n \to \infty} \frac{n^3 - n}{n^3} = \lim_{n \to \infty} \left( 1 - \frac{1}{n^2} \right) = 1 \]
By the \( p \)-test we know \( \sum \frac{1}{n^3} \) converges. By the limit comparison test \( \sum \frac{1}{n^3 - n} \) converges.

(b) Let \( a_n = \frac{1}{4^n} \) and \( b = \frac{1}{4^{n-1}} \) then
\[ \lim_{n \to \infty} \frac{\frac{1}{4^n}}{\frac{1}{4^{n-1}}} = \lim_{n \to \infty} \frac{4^n - 1}{4^n} = \lim_{n \to \infty} \left( 1 - \frac{1}{4^n} \right) = 1 \]
By the \( p \)-test \( \sum \frac{1}{4^n} \) converges therefore by the limit comparison test \( \sum \frac{1}{4^{n-1}} \) converges.
(c) Div. because

Let \( a_n = \frac{1}{n^n} \) & \( b_n = \frac{1}{\sqrt{n+2}} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n^n}{\sqrt{n+2}} \right)
\]

\[
= \lim_{n \to \infty} \sqrt{\frac{n^n}{n+2}} = 1 \neq 0
\]

Since \( \sum \frac{1}{n^n} \) div, therefore \( \sum \frac{1}{\sqrt{n+2}} \) div.

(d) Div. Use the limit comparison test.

Let \( a_n = \frac{1}{n} \) & \( b_n = \frac{n}{n^2 - 3n + 5} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n^2 - 3n + 5}{n^2} \right) = \lim_{n \to \infty} \left( 1 - \frac{3}{n} + \frac{5}{n^2} \right) = 1 \neq 0
\]

Since \( \sum \frac{1}{n} \) div, therefore \( \sum \frac{n}{n^2 - 3n + 5} \) div.

(e) Div. because \( \sum \frac{n+1}{n^2+n} = \sum \frac{n+1}{n(n+1)} = \sum \frac{1}{n} \).

(f) Div. because use the limit comparison test. Let \( a_n = \frac{1}{n} \) & \( b_n = \frac{n^3 + 3n}{2n^2 + n - 2} \)

\[
\lim_{n \to \infty} \left( \frac{n^3 + 3n}{2n^2 + n - 2} \right) = \lim_{n \to \infty} \left( \frac{n^2 + 3n}{2n^2 + n - 2} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{1 + \frac{2n}{n^2}}{2 + \frac{n}{n^2} - \frac{2}{n^2}} \right) = \frac{1}{2}
\]

Since \( \sum \frac{1}{n} \) div, therefore \( \sum \frac{2n^2 + n - 2}{n^3 + 3n} \) div.

(g) Div. Use the limit comparison test with \( a_n = \frac{1}{n} \) & \( b_n = \frac{n}{2n^2 + 1} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{\frac{n+1}{n^2}}{\frac{5n^2}{n^2}} \right) = \lim_{n \to \infty} \left( \frac{n+1}{n^2} \right) = 0
\]

Since \( \sum \frac{1}{n} \) div, therefore \( \sum \frac{n^2 + n}{n^2 + 1} \) div.
Let \( B = \sum_{k=M}^{\infty} (b_k) \) and \( n > M \) such that
\[
S_n = a_M + a_{M+1} + \ldots + a_n = \sum_{k=M}^{n} (a_k)
\]
The sequence of partial sums \((S_n)\) is bounded by \( B \) because
\[
S_n = a_M + a_{M+1} + \ldots + a_n \leq b_M + b_{M+1} + \ldots + b_n
\]
\[
\leq \sum_{k=M}^{\infty} (b_k) = B
\]
By the def. of partial sums, \( S_n, \) and given \( a_k \geq 0 \) for all \( n > M \), therefore \((S_n)\) is an increasing sequence. Since \((S_n)\) is a bounded increasing sequence it converges.

Let \( A = \lim_{n \to \infty} (S_n) \) then
\[
A = \lim_{n \to \infty} \sum_{k=M}^{n} (a_k) = \sum_{k=M}^{\infty} (a_k)
\]
Consider the infinite series we are trying to prove converges:
\[
\sum_{k=1}^{\infty} (a_k) = (a_1 + a_2 + \ldots + a_{M-1}) + \sum_{k=M}^{\infty} (a_k)
\]
\[
= a_1 + a_2 + \ldots + a_{M-1} + A
\]
Hence \( \sum_{k=1}^{\infty} (a_k) \) converges because we have a finite number of terms + a real number \( A \).
5. (a) We have
\[ \sum n^{-n} = \sum \frac{1}{n^n} \]
\[ = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots \]
\[ = 1 + \frac{1}{2^2} + \sum_{n=3}^{\infty} \frac{1}{n^n} \]

\[ \forall n \geq 3 \text{ we have} \]
\[ \frac{1}{n^n} \leq \frac{1}{n^2} \]

By the p-test, \( \sum \frac{1}{n^2} \) converges because \( p = 2 \). By the result of Q4 we have
\[ \sum n^{-n} \text{ converges.} \]

(b) By the properties of the \( \ln \) function, we know
\[ \ln(n) \geq 1 \text{ for } n \geq 3 \]

Hence
\[ \sum_{n=2}^{\infty} \left( \frac{1}{n^2 \ln(n)} \right) = \frac{1}{2^2 \ln(2)} + \sum_{n=3}^{\infty} \left( \frac{1}{n^2 \ln(n)} \right) \]

For \( n \geq 3 \)
\[ n^2 \ln(n) \geq n^2 \]
\[ \frac{1}{n^2 \ln(n)} \leq \frac{1}{n^2} \]

We know \( \sum \frac{1}{n^2} \) converges. Hence, by the result of Q4 we have
\[ \sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)} \text{ converges.} \]
Solution to 4.6

(a) $\ln(n) < n$

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

$$\sum \frac{1}{n} \text{ div } \Rightarrow \sum \frac{1}{\ln(n)} \text{ div.}$$

(b) $\sin(n) \leq n$

$$\frac{1}{\sin(n)} \geq \frac{1}{n}$$

$\sum \frac{1}{\sin(n)} \text{ conv.}$

(c) $\cos(n) \leq 1$

$$\frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$$

$\sum \frac{1}{n^2} \text{ conv. Therefore by the comparison test } \sum \frac{\cos(n)}{n^2} \text{ conv.}$

(d) $\sin(n) \leq 1$

$$\frac{\sin(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}} \text{ conv. Hence by the comparison test } \sum \frac{\sin(n)}{n^{3/2}} \text{ conv.}$

(e) $\ln(n) > 1$ for $n \geq 3$

$$\frac{\ln(n)}{n} > \frac{1}{n}$$

Since $\sum \left(\frac{1}{n}\right) \text{ div }$ therefore by the comparison test $\sum \frac{\ln(n)}{n} \text{ div.}$

(f) $\text{conv}$ because

$$e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + ...$$

$$\frac{1}{e^n} < \frac{2^n}{n^2}$$

Since $\sum ^{2^n}$ conv. Therefore $\sum \frac{1}{e^n} \text{ conv.}$
(g) **Div** because
\[
\sqrt{n^2 + 1} < \sqrt{(n+1)^2} = n+1
\]
\[
\frac{1}{n^2 + 1} > \frac{1}{n+1}
\]
\[
\sum \left(\frac{1}{n^2 + 1}\right) \text{ diverges by the comparison test: } \sum \frac{1}{n+1} \text{ div.}
\]

(h) **Div** because
\[
\frac{1}{n^2 - 3n+1} = \frac{(n-1)^2}{n-1} = n-1
\]
\[
\sum \frac{1}{n^2 - 3n+1} \geq \sum \frac{1}{n-1} = 1
\]
\[
\sum \left(\frac{1}{n-1}\right) \text{ diverges by the comparison test: } \sum \frac{1}{n-1} \text{ div.}
\]

(i) **Div**.
\[
\sqrt{1+n^2} - n < \sqrt{n^2 - 2n+1} - n
\]
\[
= \sqrt{(n-1)^2} - n = n-1 - n = -1
\]
\[
\sum (-1) \text{ diverges therefore by the comparison test: } \sum \left(\frac{1}{n^2 - n}\right) \text{ div.}
\]
7. We have

\[ a_n \leq |a_n| \]

We are given that \( \Sigma (a_n) \) converges. Therefore, by the comparison test \( \Sigma a_n \) converges.

8. We apply the limit comparison test with \( a_n = a_n \) and \( b_n = \frac{1}{n} \):

\[
\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left( \frac{a_n}{1/n} \right) = \lim_{n \to \infty} (n a_n) = L \neq 0
\]

\( \Sigma \frac{1}{n} \) is the harmonic series and diverges. By the limit comparison test \( \Sigma a_n \) diverges.

9. We are given that \( \Sigma b_n \) is convergent. Therefore \( \Sigma M b_n \) is also convergent. Since

\[ a_n \leq M b_n \] for \( n \geq K \)

then by the result of 8, \( \Sigma (a_n) \) converges.

10. (i) By hint

\[ a_n - 2 \sqrt{a_n b_n} + b_n = \left( \sqrt{a_n} - \sqrt{b_n} \right)^2 \geq 0 \]

\[ a_n + b_n \geq 2 \sqrt{a_n b_n} \geq \sqrt{a_n} \sqrt{b_n} \]

\[ \sqrt{a_n} \sqrt{b_n} \leq a_n + b_n \]

We are given that \( \Sigma (a_n b_n) \) is convergent. Therefore by the comparison test \( \Sigma \sqrt{a_n b_n} \) is convergent.

(ii) By applying the arithmetic-geometric mean inequality we have

\[ \sqrt{a_n a_{n+1}} \leq \frac{1}{2} (a_n + a_{n+1}) \leq a_n + a_{n+1} \]

We are given that \( \Sigma a_n \) is convergent, so \( \Sigma a_{n+1} \) is also convergent. By the comparison test \( \Sigma a_n a_{n+1} \) is convergent.
11. (a) By properties of $\ln$ we know that $x \geq \ln(x)$ for $x \geq 0$.

You can prove this by the corollary of MVT. Let

$$f(x) = x - \ln(x)$$

$$f'(x) = 1 - \frac{1}{x} > 0 \quad \text{for} \ x > 0$$

This means that $f(x)$ is a strictly increasing function on

$$x - \ln(x) > 0$$

$$x > \ln(x)$$

Applying this inequality to our series we have

$$\ln(a_n) \leq a_n$$

By the comparison test $\sum \ln(a_n)$ converges.

(b) Similarly for $x > 0$

$$e^x > x$$

(The infinite series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$)

Hence $a_n < e^{a_n}$. Since $\sum a_n$ diverges, by the comparison test $\sum e^{a_n}$ diverges.
12. We are given $\sum a_n$ converges. This means

$$\lim_{n \to \infty} (a_n) = 0$$

\(\exists K \in \mathbb{N}\) such that \(\forall n \geq K, a_n < 1\). Why?

If \(\forall n \in \mathbb{N}\) \(a_n > 1\) then

$$\lim_{n \to \infty} (a_n) > \lim_{n \to \infty} (1) = 1$$

This contradicts that \(\lim_{n \to \infty} (a_n) = 0\). For \(n > K\), \(a_n < 1\) therefore

$$a_n^+ < a_n$$

for \(n > K\)

By result of 04, we have \(\sum (a_n)^+\) converges.

13. Take the case of \(a_n = \frac{1}{n^2}\) then \(\sum \frac{1}{n^2}\) converges but

$$\sum a_n = \sum \frac{1}{n^2}$$

diverges.

14. We are given \(\lim_{n \to \infty} \left(\frac{a_n}{bn}\right) = L\). Therefore \(\exists K \in \mathbb{N}\) such that

\(\forall n \geq K\)

$$\frac{1}{2} L \leq \frac{a_n}{bn} \leq 2L$$

$$\frac{1}{2} Lbn < a_n < 2L bn$$

\(\sum a_n\) converges then by the comparison test \(\sum \frac{1}{2} Lbn\) converges.

This implies \(\frac{1}{2} L \sum (bn)\) converges therefore \(\sum bn\) converges.

If \(\sum bn\) converges then \(\sum (2Lbn)\) converges \(\Rightarrow \sum a_n\) converges.