Solutions to Exercise 2(a)

1. (a) \[ \sum_{n=1}^{\infty} n \]

(b) \[ 2 + 4 + 6 + 8 + \ldots \] is the sum of even numbers. This can be written as \[ \sum_{n=1}^{\infty} 2n \]

(c) \[ 1 + 3 + 5 + 7 + \ldots \] is an odd number. We can write this as \[ \sum_{n=1}^{\infty} (2n-1) \]

(d) This is an alternating series:
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \]

(e) What do you notice about each term \( b_n \) in
\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots \]?

The denominator is \( 3^n \). Hence
\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \sum_{n=0}^{\infty} \frac{1}{(3)^n} \]

(f) What is the pattern in the sequence
\[ \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots ? \]

Each term is \( \frac{2}{3} \) the previous term. We can write
\[ \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \]
2. (a) $\sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n$ is a geometric series with ratio $r = \frac{1}{3}$. Using

$$S_\infty = \frac{a}{1-r} \text{ provided } |r|<1$$

we have

$$\sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

(b) Similarly we have with common ratio $r = \frac{1}{4}$:

$$\sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

(c) We can rewrite $\frac{1}{\pi^n}$ as $\left( \frac{1}{\pi} \right)^n$. Hence

$$\sum_{n=1}^{\infty} \left( \frac{1}{\pi} \right)^n = \frac{1/\pi}{1-1/\pi} = \frac{1}{\pi-1}$$

(d) Similarly

$$\sum_{n=1}^{\infty} \left( \frac{1}{m} \right)^n = \frac{1}{m} + \left( \frac{1}{m} \right)^2 + \left( \frac{1}{m} \right)^3 + \left( \frac{1}{m} \right)^4 + \ldots$$

$$= \frac{1/\pi}{1-1/\pi} \text{ because } \frac{1}{\pi} < 1$$

$$= \frac{1}{m-1}$$
3. (a) Expanding out the given series:

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2^{2n-1}} \right) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots
\]

This is a geometric series with \(a = \frac{1}{2}\) and common ratio \(r = \frac{1}{4}\). Since \(|r| = \frac{1}{4} < 1\), we use

\[
S_{\infty} = \frac{a}{1-r}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2^{2n-1}} \right) = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{\frac{1}{2}}{3/4} = \frac{2}{3}
\]

(b) Writing out the given series:

\[
\sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n = \frac{3}{2} + \left( \frac{3}{2} \right)^2 + \left( \frac{3}{2} \right)^3 + \cdots
\]

This is a geometric series with ratio \(r = \frac{3}{2} > 1\). This series diverges.

(c) Similarly

\[
\sum_{n=1}^{\infty} e^n = e + e^2 + e^3 + e^4 + \cdots
\]

Geometric series with \(r = e > 1\). Series diverges.

(d) Expanding the given series

\[
\sum_{n=1}^{\infty} 10 \left( \frac{1}{3} \right)^n = 10 \left( \frac{1}{3} \right) + 10 \left( \frac{1}{3} \right)^2 + 10 \left( \frac{1}{3} \right)^3 + \cdots
\]

This is a geometric series with \(a = \frac{10}{3}\) and \(r = \frac{1}{3} < 1\). Applying the formula

\[
S_{\infty} = \frac{a}{1-r}
\]

we have

\[
\sum_{n=1}^{\infty} 10 \left( \frac{1}{3} \right)^n = \frac{10/3}{1-1/3} = \frac{10/3}{2/3} = 5
\]
4. This time we have not been given a formula. In each case we need to write the formula down first.

(a) Dividing the first two terms \( \frac{8}{4} = \frac{1}{2} \). The common ratio is \( r = \frac{1}{2} < 1 \) and the first term is 8. Hence

\[ 8 + 4 + 2 + 1 + \ldots = \frac{8}{1 - \frac{1}{2}} = 16 \]

(b) In the series \( 3 + 6 + 12 + 24 + \ldots \) each term is double the preceding term. We can write the series as

\[ 3 + 6 + 12 + 24 + \ldots = \sum_{n=0}^{\infty} 3(2)^n \]

\( \sum_{n=0}^{\infty} 3(2)^n \) is a geometric series with the first term \( a = 3 \) and common ratio \( r = 2 \). Because \( |r| = 2 > 1 \) therefore by (2.3) the series diverges.

(c) In the series

\[ 16 + 12 + 9 + \frac{27}{4} + \ldots \]

each term is \( 3/4 \) the preceding term \( \left( \frac{16}{12} = \frac{3}{2} , \frac{12}{16} = \frac{3}{4} \right) \).

We have

\[ 16 + 12 + 9 + \frac{27}{4} + \ldots = \sum_{n=0}^{\infty} 16 \left( \frac{3}{4} \right)^n \]

This is a geometric series with first term \( a = 16 \) and \( r = \frac{3}{4} < 1 \).

\[ 16 + 12 + 9 + \frac{27}{4} + \ldots = \frac{16}{1 - \frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64 \]
(a) The given series
\[ \sum_{n=1}^{\infty} \frac{1}{x^n} = \sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n \]
which is a geometric series with \(a = \frac{1}{x}\) and \(r = \frac{1}{x}\). Since
\[ |r| = \left|\frac{1}{x}\right| = \frac{1}{|x|} < 1 \text{ because } |x| > 1 \]
the series converges. Using
\[ S = \frac{a}{1-r} \]
we have
\[ \sum_{n=1}^{\infty} \frac{1}{x^n} = \frac{1/x}{1-1/x} = \frac{1}{x-1} \]
(Multiplying numerator and denominator by \(x\))

(b) We have
\[ \sum_{n=1}^{\infty} \left(\frac{x}{2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n \]
This is a geometric series with \(a = \frac{x}{2}\) and \(r = \frac{x}{2}\). Since
\[ |r| = \left|\frac{x}{2}\right| < \frac{1}{2} = 1 \]
the series converges. Using \(S = \frac{a}{1-r}\) we have
\[ \sum_{n=1}^{\infty} \frac{x^n}{2^n} = \frac{x/2}{1-2x/2} = \frac{x}{2-x} \]
(Multiplying top)

(c) Expanding the given series we have
\[ \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} + \cdots \]
This is a G S with \(a = \frac{1}{1+x}\) and \(r = \frac{1}{1+x}\). We are given
\[ |r| = \left|\frac{1}{1+x}\right| = \frac{1}{|1+x|} < 1 \text{ because } x > 0 \]
Applying \( S_\infty = \frac{a}{1-r} \) we have

\[
\sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{\frac{1}{1+x}}{1-\frac{1}{1+x}}
\]

\[
= \frac{1}{1+x-1} = \frac{1}{x}
\]

(multiplying numerator & denominator by \( 1+x \))

(a) Similarly we have

\[
\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \sum_{n=1}^{\infty} \left( \frac{1}{1+x^2} \right)^n
\]

\[
= \frac{1}{1+x^2} + \left( \frac{1}{1+x^2} \right)^2 + \left( \frac{1}{1+x^2} \right)^3 + \ldots
\]

Geometric series with \( a = \frac{1}{1+x^2} \) & \( r = \frac{1}{1+x^2} \)

\(|r| = \left| \frac{1}{1+x^2} \right| = \frac{1}{|1+x^2|} < 1 \) because \( 1+x^2 > 0 \)

Provided \( x \neq 0 \)

Using the infinite sum formula we have

\[
\sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \frac{1}{1-\frac{1}{1+x^2}}
\]

\[
= \frac{1}{1+x^2-1} = \frac{1}{x^2}
\]