## SECTION G **Principle of Mathematical Induction**

By the end of this section you will be able to

- understand the procedure for proof by induction
- construct proofs by induction

In this section we examine propositions concerning positive integers. The positive integers 1, 2, 3, 4,… are called *natural numbers* or *counting numbers*.
In this section, lower case letters will represent natural numbers.
Suppose we want to prove that a proposition *P* is true for all natural numbers *n*. We can't give separate proofs for each $n=1,\ 2,\ 3,\ \cdots$ because that would be an endless exercise.
Consider the following example:
For every natural number *n*, prove the proposition $P(n)$ given by

$$1+2+3+4+\cdots+n=\frac{1}{2}n(n+1)$$

*What does this proposition mean?*

It means that if we add the first $n$ natural numbers then the answer will be $\frac{1}{2}n(n+1)$. For example if we add the first 2 numbers we have

$$1+2=\frac{1}{2}(2)(2+1)=3 \qquad \text{[Substituting } n=2 \text{ into the above]}$$

Similarly we have

$$\underbrace{1+2+3}_{3\text{ Terms}}=\frac{1}{2}(3)(3+1)=6 \qquad \left[\begin{array}{l}\text{Adding the first 3 natural numbers,}\\ \text{that is } n=3.\end{array}\right]$$

$$\underbrace{1+2+3+4}_{4\text{ Terms}}=\frac{1}{2}(4)(4+1)=10 \qquad \left[\begin{array}{l}\text{Adding the first 4 natural numbers,}\\ \text{that is } n=4.\end{array}\right]$$

$$\underbrace{1+2+3+4+5}_{5\text{ Terms}}=\frac{1}{2}(5)(5+1)=15 \qquad \left[\begin{array}{l}\text{Adding the first 5 natural numbers,}\\ \text{that is } n=5.\end{array}\right]$$

Hence we have shown that the given proposition is true for $n=2,\ 3,\ 4$ and $5$. Verifying this proposition for $n=100$ would be a tedious task. Even if we verify the proposition for the first *million* natural numbers, this does not mean the proposition is true for all values of *n*. *So how can we prove the given proposition is true for all n?*
One way is to use mathematical induction, described below.

### G1 Principle of Mathematical Induction

**Mathematical induction** is a powerful tool used to prove propositions concerning natural numbers.

**Principle of Mathematical Induction (I.30)**

For each natural number *n*, let $P(n)$ be a proposition about *n*. If $P(n)$ satisfies:

1) $P(1)$ is true.
2) For any natural number *k*; if $P(k)$ is true then $P(k+1)$ is true.

Then for *all* natural numbers, $n$, the proposition $P(n)$ is true.

The assumption $P(k)$ is true (where $k$ is a natural number) is called the **inductive hypothesis**. We use the inductive hypothesis $P(k)$ to prove the proposition for the next number $P(k+1)$.

*By using the above 2 steps, can you see why we can conclude that $P(n)$ is true for all n?*

Step 1) says the proposition is true for $n=1$. As the proposition is true for $n=1$, so step 2) says it is true for the next number $n=1+1=2$. Repeating step 2) says the proposition is true for the next number $n=2+1=3$ and so on. Steps 1) and 2) suggest that;
$P(1)$ implies $P(2)$, $P(2)$ implies $P(3)$, $P(3)$ implies $P(4)$, $P(4)$ implies $P(5)$, and so on.
This is sometimes called the domino effect. If the first domino topples and causes the next one to topple, then we end up with *all* the dominos knocked over.

Figure 1 Domino Effect

$P(1)$ implies $P(2)$ - the 1st domino knocks over the second; $P(2)$ implies $P(3)$ - the 2nd domino knocks over the 3rd ; $\cdots$ $P(100)$ implies $P(101)$ - the 100th domino knocks over the 101th domino and so on. $P(k) \Rightarrow P(k+1)$ means that we assume the $k$ th domino falls which forces the $(k+1)$ th domino to fall. In this way *all* the dominos fall, so the proposition is knocked down or is true for all *n*.

If some of the dominos are not properly set up and one of them is *not* knocked over by the previous one then the statement $P(k) \Rightarrow P(k+1)$ is *not* shown which means the proposition $P(n)$ may be true but *cannot* be proven by mathematical induction.

Proof by mathematical induction is very useful in many branches of mathematics especially linear and abstract algebra, number theory and set theory.

The process is as follows; we show $P(1)$ is true and by assuming $P(k)$ is true we prove $P(k+1)$ is true. If *both* $P(1)$ is true *and* $P(k)$ implies $P(k+1)$, then proposition $P(n)$ is true for *all* natural numbers *n*.

To show $P(k)$ implies $P(k+1)$ we use the following 3 stages:

Stage 1 Write down the proposition for $n=k$.
Stage 2 Write down the proposition for $n=k+1$. (Our goal is to prove this result.)
Stage 3 Prove the proposition $P(k+1)$ using $P(k)$.

To prove $P(n)$ by induction we can use the following flow chart:

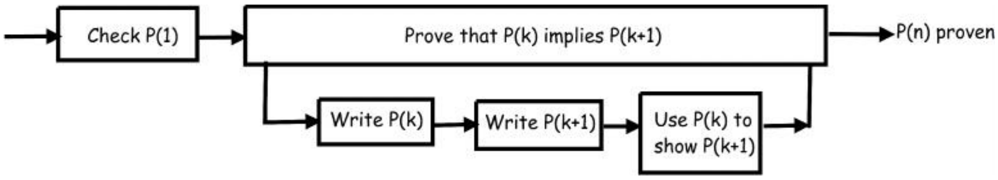


Figure 2

*Why isn't it good enough to say the proposition is true if it works for the first few numbers?*
Consider the proposition

$$P(n): \quad n^2 - n + 41 \text{ is prime for all natural numbers } n$$

$P(1)$ is true because 41 is prime. Actually the proposition $P(n)$ is true for $n = 1$, 2, 3, $\cdots$, 40. However the proposition is *not* true for $n = 41$ because

$$41^2 - 41 + 41 = 41^2 \text{ which is } \textit{not} \text{ prime}$$

Therefore proposition $P(n)$ is false. You are given a "proof" in Exercise 1(g) of this result and you need to find the error in the "proof".
Verifying $P(n)$ for 40 or more values of *n* does *not* constitute a proof.

## G2 Examples

Example 1

For every natural number *n*, prove the above proposition $P(n)$ given by

$$1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$$

We have already shown this proposition is true for $n = 2,\ 3,\ 4$ and $5$.
We need to show this result for all the natural numbers $n$. *How?*
We employ mathematical induction because the proposition concerns the natural number $n$.

*Proof*

First we check the proposition for $n = 1$:

$$1 = \frac{1}{2}(1)(1+1) \qquad \checkmark$$

Hence the proposition is true for $n = 1$.

Stage 1

Next we *assume* the given proposition is true for $n = k$, that is $P(k)$ - we assume we have knocked down the $k^{\text{th}}$ domino. This is the induction hypothesis. *How do we write this* $P(k)$*?*

By substituting $n = k$ into the given proposition

$$1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$$

which gives

$$1+2+3+4+\cdots+k = \frac{1}{2}(k)(k+1) \qquad (*)$$

Stage 2

We have labelled this result by (*) because we are going to prove the proposition for the next number $k+1$ by using (*). We use the fall of the $k^{\text{th}}$ domino to knock down the $(k+1)^{\text{th}}$ domino. *How do we write the proposition for the next number, $P(k+1)$?*

By substituting $n=k+1$ into the given proposition, $1+2+3+4+\cdots+n=\frac{1}{2}n(n+1)$:

$$1+2+3+4+\cdots+k+(k+1)=\frac{1}{2}(k+1)\underbrace{(k+1+1)}_{=k+2}=\frac{1}{2}(k+1)(k+2) \qquad (**)$$

Stage 3

This means that *we have to prove* the sum of the first $(k+1)$ natural numbers (the sum of $1,\ 2,\ 3,\cdots,k,\ k+1$) is equal to $\frac{1}{2}(k+1)(k+2)$. It is critical that you realise we need to prove (**). We have *only* stated $P(k+1)$ *not yet* proven it. The challenge is to show that the left hand side is equal to the right hand side of (**). *How?*

By examining the left hand side and simplifying the sum of first $k$ terms by using (*):

$$1+2+3+4+...+k+(k+1)=\underbrace{1+2+3+4+...+k}_{=\frac{1}{2}k(k+1)\text{ by (*)}}+(k+1)$$

$$=\frac{1}{2}k(k+1)+(k+1) \qquad [\text{Simplifying}]$$

$$=\frac{1}{2}(k+1)k+\frac{1}{2}(k+1)2 \qquad \left[\text{Rewriting } (k+1)=\frac{1}{2}(k+1)2\right]$$

$$=\frac{1}{2}(k+1)(k+2) \qquad [\text{Factorizing}]$$

The last line is the right hand side of (**). Hence our result holds by the principle of mathematical induction because we have shown (**).

■

Notice how we assume $P(k)$ to be true and then use it to prove $P(k+1)$. [Assume the $k^{\text{th}}$ domino is knocked over and it forces the $(k+1)^{\text{th}}$ domino to fall.] The proposition $P(k)$ in the above was (*) and we used this in the derivation of $P(k+1)$ which was

$$1+2+3+4+\cdots+k+(k+1)=\frac{1}{2}(k+1)(k+2) \qquad (**)$$

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$ is true so we have the required result, $1+2+3+4+...+n=\frac{1}{2}n(n+1)$, by mathematical induction.

Using this formula makes life a lot easier because, say we want to find the sum of the first 100 natural numbers then using the formula of Example 1 we have

$$[1+2+3+4+\cdots+100]=\frac{1}{2}100(101)=5050$$

Much easier than trying to add the natural numbers $1,\ 2,\ 3,\ 4,\ \cdots,\ 99$ and $100$.

Example 2

For every natural number $n$ prove the proposition $P(n)$ given by

$$1+3+5+7+\cdots+(2n-1)=n^2$$

*What does this proposition mean?*

It means that if we add the first *n odd* counting numbers then the answer will be the square of $n$. For example if we add the first 2 odd counting numbers we have

$$1+3=4=2^2 \qquad \text{[Substituting } n=2 \text{ into the given proposition]}$$

Similarly we have

$$\underbrace{1+3+5}_{\text{3 Terms}}=3^2 \qquad \text{[Adding the first 3 odd numbers, that is } n=3\text{]}$$

$$\underbrace{1+3+5+7}_{\text{4 Terms}}=4^2 \qquad \text{[Adding the first 4 odd numbers, that is } n=4\text{]}$$

$$\underbrace{1+3+5+7+9}_{\text{5 Terms}}=5^2 \qquad \text{[Adding the first 5 odd numbers, that is } n=5\text{]}$$

and so on. Again the proposition is true for $n=2, 3, 4$ and 5. We need to show this result holds for *all* the natural numbers $n$. *How?*

We apply mathematical induction.

*Proof*

First we check the proposition for $n=1$:

$$1=1^2 \quad \checkmark$$

Hence the proposition is true for $P(1)$.

Stage 1

Next we *assume* the given proposition is true for $n=k$ that is $P(k)$:

$$\underbrace{1+3+5+7+\cdots+(2k-1)}_{k \text{ terms}}=k^2 \qquad (\dagger)$$

We have labelled $P(k)$ with (†) because we are going to prove the proposition for the next number $k+1$ by using (†). (We use the $k^{\text{th}}$ domino to knock down the $(k+1)^{\text{th}}$ domino.)

Stage 2

*How do we write the proposition $P(k+1)$?*

By substituting $n=k+1$ into the given proposition, $1+3+5+7+\cdots+(2n-1)=n^2$:

$$\underbrace{1+3+5+7+...+(2k-1)}_{\text{First } k \text{ terms}}+\underbrace{(2(k+1)-1)}_{(k+1)\text{th term}}=(k+1)^2 \qquad (\dagger\dagger)$$

Stage 3

We need to prove this, (††), result. The challenge is to show that the left hand side is equal to the right hand side of (††). *How?*

We can simplify by writing the sum up to $(2k-1)$ on the left by using (†); we have

$$\underbrace{1+3+5+7+\cdots+(2k-1)}_{=k^2 \text{ by } (\dagger)}+(2k+1)=k^2+2k+1 \qquad \text{[Simplifying]}$$

$$=(k+1)^2 \qquad \text{[Factorizing]}$$

The last line is the same as the right hand side of (††). By the principle of mathematical induction we have our required result.

■

In the above example we first showed that the given result is true for $P(1)$:

$$1 = 1^2 \qquad \text{[The first domino has fallen.]}$$

Secondly we assumed it is true for $P(k)$ - the $k^{\text{th}}$ domino has fallen:

$$1+3+5+7+\cdots+(2k-1)=k^2 \qquad [n=k]$$

and finally we used this assumption to produce the result for $P(k+1)$ - the $k^{\text{th}}$ domino has knocked down the $(k+1)^{\text{th}}$ domino:

$$1+3+5+7+\cdots+(2k-1)+(2k+1)=(k+1)^2 \qquad [n=k+1]$$

The given result, $1+3+5+7+\cdots+(2n-1)=n^2$, follows by the principle of mathematical induction – we have achieved our goal of knocking down all the dominos.

Example 3

Prove for every natural number $n$ the proposition $P(n)$ given by

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

We can use mathematical induction, but the proof requires a little more algebraic manipulation than the previous two examples.

*Proof*

We first show the proposition is true for $n=1$ by substituting this into $P(n)$ :

$$1^2=\frac{1(1+1)(2+1)}{6} \qquad \checkmark$$

Therefore $P(1)$ is correct. *What do we do next?*

Assume the given proposition is true for $n=k$, that is $P(k)$:

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6} \qquad (*)$$

The ‘meat’ in mathematical induction is to show the given proposition is true for the next number $n=k+1$ by employing (*). *How?*

We first write down what *we need* to prove, that is write down the proposition $P(k+1)$ by substituting $n=k+1$ into the given proposition:

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6} \qquad (\dagger)$$

*What do we need to show?*

The left hand side is equal to the right hand side of (†). *How?*

We know by (*) that the sum up to $k^2$ is equal to $\frac{k(k+1)(2k+1)}{6}$, so we use this on the left hand side and then apply algebraic manipulation to get the right hand side:

$$\underbrace{1^2+2^2+3^2+\cdots+k^2}_{=\frac{k(k+1)(2k+1)}{6} \text{ by } (*)}+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6} \qquad \text{[Common denominator]}$$

$$=\frac{k+1}{6}\left[k(2k+1)+6(k+1)\right] \qquad \left[\text{Factorizing out } \frac{k+1}{6}\right]$$

$$=\frac{k+1}{6}\left[2k^2+\underbrace{k+6k}_{=7k}+6\right] \qquad \text{[Expanding]}$$

$$=\frac{k+1}{6}\left[2k^2+7k+6\right] \qquad \text{[Simplifying the quadratic]}$$

$$=\frac{k+1}{6}(k+2)(2k+3) \qquad \text{[Factoring the quadratic]}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

The last line is the same as the right hand side of (†). This completes our proof.

■

In the above example, we first checked that the result was true for $n=1$. Then we assumed the result was true for $n=k$. Finally using this assumption (inductive hypothesis) we proved the result was true for $n=k+1$. By mathematical induction we have proven

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

## G3 Flaws in applying Mathematical Induction

Consider the following proposition:

$P(n)$: All real numbers are equal.

This proposition is in fact a fallacy, although it's not immediately clear why;

*"Proof"*

Clearly the result is true for $n=1$ because one real number, say $x_1=x_1$.

Assume that $P(k)$ is true, that is any $k$ numbers are equal:

$$x_1=x_2=x_3=\cdots=x_k \qquad (*)$$

Required to prove $P(k+1)$ is true; that is we need to show

$$x_1=x_2=x_3=\cdots=x_k=x_{k+1} \qquad (\dagger)$$

Applying the induction hypothesis to the last $k$ numbers we have

$$x_2=x_3=\cdots=x_k=x_{k+1} \qquad (**)$$

Combining (*) and (**) gives us our result (†), that is all real numbers are equal.

*Where is the error?*

If $k=1$ then (**) only has one real number $x_2$ and (*) only has the real number $x_1$ which means there is no overlap between (*) and (**) so we *cannot* say $x_1=x_2$. Hence $P(1) \not\Rightarrow P(2)$ [does *not* imply] or the first domino does *not* knock down the second so the whole induction argument fails. Remember *all* the dominos must fall for a valid proof by induction.

SUMMARY
We use mathematical induction to prove propositions involving natural numbers.
Using the principle of mathematical induction to prove a proposition $P(n)$ involves

1. Showing the result for $n=1$, that is $P(1)$.
2. Assuming the result is true for a natural number *k*, that is assuming $P(k)$ is true.
3. Proving the result for $n=k+1$, that is proving $P(k+1)$ using result $P(k)$.