Section G **Subsequences**

By the end of this section you will be able to
- understand what is meant by a subsequence
- prove the Bolzano Weierstrass theorem

G1 **Definition of a Subsequence**

*What do you think the term **subsequence** means?*
A sequence within a sequence. Generally a subsequence is a selection of the terms in the sequence. For example:
Let $x_n = 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ then the following are subsequences of (x_n):

2, 4, 6, 8, 10, …
or 1, 3, 5, 7, 9, …..
or 2, 7, 9, 10, 11, …..

The following are **not** subsequences of (x_n):

a) 2, 1, 4, 6, 8, 10, …
b) 1, 7, 3, 5, 9, 11,…
c) 0, 2, 4, 6, 8, 10, …

Definition (5.27). Let $x_n = x_1,\ x_2,\ x_3,\ x_4,\ x_5, \cdots$ be a sequence of real numbers and if the natural numbers $n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$ are strictly increasing then

$$x_{n_k} = x_{n_1},\ x_{n_2},\ x_{n_3},\ x_{n_4},\ x_{n_5}, \cdots$$

is a **subsequence** of (x_n).

Let $x_n = x_1,\ x_2,\ x_3,\ x_4,\ x_5, \cdots$ be a sequence of real numbers then the following are subsequences of (x_n):

$x_2,\ x_4,\ x_6,\ x_8, \cdots$
or $x_1,\ x_3,\ x_5,\ x_7, \cdots$
or $x_1,\ x_4,\ x_7,\ x_{23}, \cdots$

The following are **not** subsequences of (x_n):

a) $x_2,\ x_1,\ x_5,\ x_2, \cdots$
b) $x_9,\ x_3,\ x_{1000},\ x_7, \cdots$
c) $x_{111},\ x_{12},\ x_7,\ x_{10}, \cdots$

*Why are these **not** subsequences of (x_n) ?*

Because the subscript numbers in parts a), b) and c) are **not** strictly increasing as you down the terms in the sequence. For example the first term in part a) is x_2 which is the second term in the given sequence $x_n = x_1,\ x_2,\ x_3,\ x_4,\ x_5, \cdots$. The next term is x_1 in part a) which is the first term in the main sequence (x_n). *Why isn't the sequence in part b) a subsequence of (x_n) ?*
Because the first term, x_9, is the 9th term of the given sequence and the second term, x_3, is the third term of the given sequence. Additionally the 3rd term, x_{1000}, is the

thousandth term of the given sequence (x_n) but 4th term, x_7, is the seventh term of (x_n).

The subscript numbers need to be strictly increasing of the subsequence. *What does this mean?*

Means that you have to move **down** the sequence. Consider the following example:

Let $x_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$ then the following are subsequences of (x_n):

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \cdots$$
$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$$
$$1, \frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \cdots$$

The following are **not** subsequences of (x_n):

a) $\frac{1}{5}, \frac{1}{4}, \frac{1}{100}, \frac{1}{8}, \frac{1}{10}, \cdots$

b) $1, 0, \frac{1}{4}, 0, \frac{1}{16}, \cdots$

c) $0, \frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \cdots$

Why isn't part a) a subsequence of $x_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$?

The first term, $\frac{1}{5}$, is 5th term of (x_n) and the second term, $\frac{1}{4}$, is 4th term of (x_n).

This subsequence is moving **up** the main sequence (x_n).

Why isn't the above part b) a subsequence of $x_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$?

Because there is **no** 0 in the original sequence (x_n).

Similarly part c) is not a subsequence of (x_n) because there is **no** 0 in the given sequence (x_n).

G2 **Bolzano Weierstrass Theorem**

Peak Term (5.28). We say the term x_M of the sequence (x_n) is the **peak term** if

$$x_M \geq x_n \quad \text{for all } n \geq M$$

This means that x_M is greater than or equal to all terms that follow x_M.

Monotone Subsequence Theorem (5.29).

Let (x_n) be a sequence of real numbers then (x_n) has a subsequence that is monotone.

Proof.

We consider two cases:

1) The sequence (x_n) has **infinitely** many peaks.
2) The sequence (x_n) has **finitely** many peaks.

Case 1: Assume sequence $\left(x_n\right)$ has infinitely many peaks. We can list the peaks by increasing subscripts

$$x_{M_1},\ x_{M_2},\ x_{M_3},\ \cdots, x_{M_k},\ \cdots$$

Because each term is a peak we have

$$x_{M_1} \geq x_{M_2} \geq x_{M_3} \geq \cdots \geq x_{M_k} \geq \cdots$$

Therefore the subsequence $\left(x_{M_k}\right)$ of peaks is a decreasing (monotone) sequence of $\left(x_n\right)$.

Case 2: Assume sequence $\left(x_n\right)$ has finitely many peaks. . We can list the peaks by increasing subscripts

$$x_{M_1},\ x_{M_2},\ x_{M_3},\ \cdots, x_{M_r}$$

Let $s_1 = M_r + 1$ be the first index after the last peak. Since x_{s_1} is not a peak, there exists

$$s_2 > s_1$$

such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists

$$s_3 > s_2$$

such that $x_{s_2} < x_{s_3}$. Continuing in this way we get an increasing (monotone) sequence $\left(x_{s_k}\right)$ of the given sequence $\left(x_n\right)$.

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Bolzano Weiestrass Theorem (5.30).
A bounded sequence of real numbers has a convergent subsequence.

Proof.

Let $\left(x_n\right)$ be a sequence of real numbers that are bounded. By the above proposition (5.29) the sequence $\left(x_n\right)$ has a subsequence that is monotone. This subsequence is bounded. Hence by the "Monotone Convergence Theorem" (5.23) the subsequence is convergent.

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(5.23) Every bounded monotone sequence is convergent.