SECTION F **Applications of Eigenvalues and Eigenvectors to Quadratic Forms**

By the end of this section you will be able to

- find a matrix given a quadratic form
- convert a quadratic into diagonal form
- decide whether a matrix is positive, negative or indefinite

You might be surprised that quadratic forms which involve expressions of degree 2 are covered in linear algebra because in essence linear algebra is the study of linear equations. In this section you will find that linear algebra is a useful tool to describe quadratic forms. You will need to recall your work from section D regarding the diagonalization of symmetric matrices.

F1 Introduction to Quadratic Forms

What does the term quadratic mean?

It is an expression of second degree. For example a *quadratic polynomial* is a polynomial such as ax^2+bx+c. An expression of the form

(7.35) $$ax^2+2hxy+by^2 \qquad [a, b \text{ and } h \text{ are real numbers}]$$

is called a *quadratic form* in two variables *x* and *y*. Of course we can have a quadratic form of more than two variables as well. In this section we place quadratic forms into matrix form as the examples below show. Putting a quadratic into matrix form means we can use matrices to solve quadratic equations.

Quadratic forms are also used to find where a function of more than one variable is a maximum or minimum.

Example 37

Let $\mathbf{A}=\begin{pmatrix}-2 & 4\\ 4 & 3\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$. Determine $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Solution

What is $\mathbf{x}^T$ *equal to?*

Well $\mathbf{x}^T$ is the vector **x** transposed. That is $\mathbf{x}^T=\begin{pmatrix}x & y\end{pmatrix}$. Thus we have

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &= \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}-2 & 4\\ 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\ &= \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}-2x+4y\\ 4x+3y\end{pmatrix}\\ &= \left(x[-2x+4y]+y[4x+3y]\right)\\ &= -2x^2+\underbrace{4xy+4xy}_{=8xy}+3y^2=-2x^2+8xy+3y^2\end{aligned}$$

Note that $\mathbf{x}^T\mathbf{A}\mathbf{x} = -2x^2+8xy+3y^2$ is an example of a polynomial in quadratic form.

The matrix **A** is called the *matrix of quadratic form*.

The following is *not* a quadratic form (of degree 2) in two variables *x* and *y*:

$$x^2+x^2y-y^2$$

Why not?

Because the middle term $x^2y \ (= xxy)$ is cubic (of degree 3) in nature.

Quadratic form is given by $\mathbf{x}^T\mathbf{Ax}$ where **A** is a *symmetric* matrix. We carry out the following operations:

$\boxed{\text{Vector } \mathbf{x}^T}$ X $\boxed{\text{Symmetric matrix } \mathbf{A}}$ X $\boxed{\text{Vector } \mathbf{x}}$ gives quadratic form $\mathbf{x}^T\mathbf{Ax}$

Figure 1

Quadratic forms are used in physics and engineering. For example the kinetic energy, *ke*, of a rigid body is given by

$$ke = \frac{1}{2}\mu^T \mathbf{A}\mu$$

where **ω** is the angular velocity vector and **A** is a 3 by 3 symmetric matrix.

Another application of quadratic form is finding the length $\|\mathbf{Ax}\|$ of the vector $\mathbf{Ax}$:

$$\|\mathbf{Ax}\|^2 = (\mathbf{Ax})^T \mathbf{Ax} = \mathbf{x}^T \left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}$$

Matrix **A** may not be symmetric so we use the symmetric matrix $\mathbf{A}^T\mathbf{A}$ to find the length.

We can also use quadratic forms to find the norm of a matrix. The norm of a matrix **A** tells us how much the matrix **A** magnifies a given vector **x** when applied to it.

For most applications the norm of any matrix **A** denoted by $\|\mathbf{A}\|$ is defined as

$$\|\mathbf{A}\|^2 = \max\left[\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}\right] \quad \text{provided } \|\mathbf{x}\| = 1$$

[In Matlab the command `norm(A)` evaluates this norm of the matrix **A**.]

The function max is the maximum value of the quadratic form $\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}$ for all unit vectors **x**.

Example 38

Find the quadratic form $\mathbf{x}^T\mathbf{Ax}$ of the matrix $\mathbf{A} = \begin{pmatrix} -3 & -5 \\ -5 & 4 \end{pmatrix}$.

<u>Solution</u>

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $\mathbf{x}^T = \begin{pmatrix} x & y \end{pmatrix}$ and we have

$$\begin{aligned} \mathbf{x}^T\mathbf{Ax} &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -3 & -5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -3x-5y \\ -5x+4y \end{pmatrix} \\ &= \left(x[-3x-5y]+y[-5x+4y]\right) = -3x^2-10xy+4y^2 \end{aligned}$$

We have $\mathbf{x}^T\mathbf{Ax} = -3x^2 - 10xy + 4y^2$.

What do you notice about the matrix **A** *in both the above examples?*

We have $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -3 & -5 \\ -5 & 4 \end{pmatrix}$. In both cases the matrix is *symmetrical*, that is the transpose of **A** is **A**; $\mathbf{A}^T = \mathbf{A}$.

In the next example we derive the general quadratic form.

Example 39

Let $\mathbf{A} = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ where *a*, *b* and *h* are real numbers and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Determine $\mathbf{x}^T\mathbf{Ax}$.

<u>Solution</u>

Note that **A** is a symmetric matrix. We have $\mathbf{x}^T = \begin{pmatrix} x & y \end{pmatrix}$ therefore

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} ax+hy \\ hx+by \end{pmatrix} \\ &= \left(x\left[ax+hy\right]+y\left[hx+by\right]\right) \\ &= ax^2+2hxy+by^2\end{aligned}$$

What do you notice about $ax^2 + 2hxy + by^2$ *?*

This is the above quadratic form given in (7.35). Thus we conclude that the quadratic form:

$$\boxed{a}x^2 + 2\boxed{h}xy + \boxed{b}y^2 = \mathbf{x}^T\begin{pmatrix} \boxed{a} & \boxed{h} \\ \boxed{h} & \boxed{b} \end{pmatrix}\mathbf{x} \quad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We can extend quadratic form to any number of variables. We write a quadratic form in n variables $x_1,\ x_2,\ x_3,\ \cdots$ and x_n by using an n by n matrix as the following definition states:

Definition (7.36). Let **A** be an n by n symmetric matrix. The function f given by

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is called the *quadratic form* in n variables $x_1,\ x_2,\ x_3,\ \cdots$ and x_n.

Example 40

Find the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ for the symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 \\ 4 & 2 & 5 \\ 7 & 5 & 3 \end{pmatrix}$.

<u>Solution</u>

Let $\mathbf{x}^T = \begin{pmatrix} x & y & z \end{pmatrix}$ then we have

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 4 & 7 \\ 4 & 2 & 5 \\ 7 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} x+4y+7z \\ 4x+2y+5z \\ 7x+5y+3z \end{pmatrix} \\ &= \left(x\left[x+4y+7z\right]+y\left[4x+2y+5z\right]+z\left[7x+5y+3z\right]\right) \\ &= x^2+4xy+7xz+4xy+2y^2+5yz+7xz+5yz+3z^2 \\ &= x^2+8xy+14xz+2y^2+10yz+3z^2 \qquad \left[\text{Collecting like terms}\right]\end{aligned}$$

Note that $\mathbf{x}^T\mathbf{A}\mathbf{x} = x^2 + 8xy + 14xz + 2y^2 + 10yz + 3z^2$ is a quadratic form in 3 variables x, y and z.

Next we derive the general quadratic form in three variables.

Example 41

Let $\mathbf{A}=\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Determine $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Solution

We have $\mathbf{x}^T = \begin{pmatrix} x & y & z \end{pmatrix}$. Therefore

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{pmatrix} \\ &= \left(x[ax+dy+ez]+y[dx+by+fz]+z[ex+fy+cz]\right) \\ &= ax^2+dyx+ezx+dxy+by^2+fzy+exz+fyz+cz^2 \\ &= ax^2+2dyx+2ezx+by^2+2fzy+cz^2 \qquad [\text{Collecting like terms}]\end{aligned}$$

Thus the general quadratic form in three variables *x*, *y* and *z* is given by

$$ax^2 + 2d(yx) + 2e(zx) + by^2 + 2f(zy) + cz^2 = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

where $\mathbf{A}=\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ is a symmetric matrix and $\mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is equivalent to

$$(7.37) \quad ax^2 + 2\boxed{d}(xy) + 2\boxed{e}(zx) + by^2 + 2\boxed{f}(zy) + cz^2 = \mathbf{x}^T\begin{pmatrix} a & \boxed{d} & \boxed{e} \\ \boxed{d} & b & \boxed{f} \\ \boxed{e} & \boxed{f} & c \end{pmatrix}\mathbf{x}$$

F2 Finding the Matrix given the Quadratic Form

The other way round of finding the matrix given the quadratic form is more difficult. Examining the matrix in the above formula (7.37), we have the entries on the leading diagonal give the coefficients of x^2, y^2 and z^2 respectively. The *d* entry of the matrix in the above (7.37) is *half* the *xy* coefficient because the *xy* coefficient is $2d$. Similarly the *e* entry is *half* the *zx* coefficient and the *f* entry is *half* the *zy* coefficient.

The coefficients of the single variable like x^2, y^2 and z^2 lie on the leading diagonal. Halving the coefficients of mixed variables such as xy, xz and yz gives the entries to the right and left of the leading diagonal.

Example 41

Write the quadratic form $2x^2+7yx+3zx-8y^2-2zy-3z^2$ as $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Solution

Using the above (7.37) with $a=2$, $d=\frac{7}{2}$, $e=\frac{3}{2}$, $b=-8$, $f=-\frac{2}{2}=-1$ and $c=-3$:

$$2x^2+7(yx)+3(zx)-8y^2-2(zy)-3z^2 = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 2 & 7/2 & 3/2 \\ 7/2 & -8 & -1 \\ 3/2 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Example 42

Write the quadratic form $-2x^2+7xy-8y^2$ as $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

<u>Solution</u>

Since we have 2 variables *x* and *y* so the matrix is of size 2 by 2. The entries on the leading diagonal are the coefficients of x^2 and y^2 respectively. Both entries along the other diagonal is $7/2$ because the *xy* coefficient is 7. Thus we have

$$-2x^2+7xy-8y^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -2 & 7/2 \\ 7/2 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 43

Convert the quadratic form $x^2+y^2+z^2$ to matrix form $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

<u>Solution</u>

What size matrix will we have this time?

We have three variables *x*, *y* and *z* therefore the matrix size is 3 by 3. The coefficients of x^2, y^2 and z^2 is 1 therefore *all* the leading diagonal entries are 1. There are *no* mixed *xy*, *xz* and *yz* terms therefore the remaining entries in the matrix are zero. We have

$$x^2+y^2+z^2=\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x & y & z \end{pmatrix}\mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The 3 by 3 identity matrix **I** is the matrix of the quadratic form $x^2+y^2+z^2$ or we can say

$$\mathbf{x}^T\mathbf{I}\mathbf{x} \;=\; \mathbf{x}^T\mathbf{x} \;=\; x^2 \;+\; y^2 \;+\; z^2$$

This quadratic form $\mathbf{x}^T\mathbf{I}\mathbf{x}$ gives the length or norm of the vector **x**.

F3 Converting to Diagonal form

In this subsection we change the quadratic form in variables $x_1,\ x_2,\ x_3,\ \cdots$ and x_n into new variables $y_1,\ y_2,\ y_3,\ \cdots$ and y_n. *Why?*

We will find it simpler to deal with these new variables because we convert the given quadratic form into a more easy to use format called *diagonal form*.

Remember we discussed in section D that it is easier to deal with a diagonal matrix. We convert a given quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ into a diagonal format $\mathbf{y}^T\mathbf{D}\mathbf{y}$ where **D** is a diagonal matrix. This $\mathbf{y}^T\mathbf{D}\mathbf{y}$ is called **diagonal form**.

What is the point of converting the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ *to* $\mathbf{y}^T\mathbf{D}\mathbf{y}$ *?*

Well $\mathbf{y}^T\mathbf{D}\mathbf{y}$ contains *no* mixed or cross product terms, that is terms of the sort y_1y_2, y_1y_3, y_2y_3 … are *not* part of $\mathbf{y}^T\mathbf{D}\mathbf{y}$. Thus we can show that $\mathbf{y}^T\mathbf{D}\mathbf{y}$ only contains terms of the form $y_1^{\,2},\ y_2^{\,2},\ y_3^{\,2},\ \cdots$ and $y_n^{\,2}$. Actually $\mathbf{y}^T\mathbf{D}\mathbf{y}$ is a sum of squares.

The next example illustrates this.

Example 44

Write the quadratic form $10x_1^{\,2}-8x_1x_2+4x_2^{\,2}$ as $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Find the orthogonal matrix **Q** which diagonalizes **A** and the diagonal matrix **D** such that $\mathbf{Q}^T\mathbf{A}\mathbf{Q}=\mathbf{D}$. Also determine the diagonal form $f(\mathbf{y})=\mathbf{y}^T\mathbf{D}\mathbf{y}$ where $\mathbf{y}=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

<u>Solution</u>

What are the leading diagonal entries in the matrix **A** *of quadratic form* $10x_1^2 - 8x_1x_2 + 4x_2^2$*?*
10 and 4 because these are the coefficients of x_1^2 and x_2^2 respectively. *What are the other entries in the matrix* **A***?*
-4 because half of the cross product x_1x_2 coefficient (-8) is equal to -4. Thus we have

$$10x_1^2 - 8x_1x_2 + 4x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix $\mathbf{A} = \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}$ is a symmetric matrix which means we can orthogonally diagonalize this matrix by finding the eigenvalues and corresponding normalised eigenvectors. This was covered in section D and we will assume that you can verify the following eigenvalues and eigenvectors belonging to these eigenvalues in your own time:

$$\lambda_1 = 2,\ \mathbf{u} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \lambda_2 = 12,\ \mathbf{v} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

What is the eigenvector matrix **Q** *equal to?*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad \left[\text{Remember } \mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}\right]$$

What is the eigenvalue (diagonal) matrix **D** *equal to?*
The matrix with the leading diagonal entries being the eigenvalues of matrix **A**:

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \qquad \begin{bmatrix} \text{Because 2 and 12 are} \\ \text{the eigenvalues of } \mathbf{A} \end{bmatrix}$$

How do we find $f(\mathbf{y}) = \mathbf{y}^T\mathbf{D}\mathbf{y}$*?*

$$\begin{aligned} f(\mathbf{y}) = \mathbf{y}^T\mathbf{D}\mathbf{y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} 2y_1 \\ 12y_2 \end{pmatrix} \\ &= 2y_1^2 + 12y_2^2 \end{aligned}$$

What is the advantage of converting the given quadratic form into diagonal form?
Well if we examine the above example we converted from the given quadratic to a sum of squares:

$$10x_1^2 - 8x_1x_2 + 4x_2^2 \quad \Rightarrow \quad 2y_1^2 + 12y_2^2$$

This new diagonal form is easier to graph as will be shown in the next section. Also the solution to the equation $10x_1^2 - 8x_1x_2 + 4x_2^2 = c$ is equivalent to solving the easier equation

$$2y_1^2 + 12y_2^2 = c$$

In general a complicated problem can been reduced to something more manageable by using diagonal form.
Next we check the above result is correct by going from the *y* back to the *x* variable.

Example 45
Check that $f(\mathbf{y}) = 2y_1^2 + 12y_2^2 = 10x_1^2 - 8x_1x_2 + 4x_2^2$.

Solution
From Proposition (7.38) we have $\mathbf{Q}\mathbf{y} = \mathbf{x}$. Taking inverse **Q** of both sides of $\mathbf{Q}\mathbf{y} = \mathbf{x}$ gives

$$\mathbf{y} = \mathbf{Q}^{-1}\mathbf{x} = \mathbf{Q}^T\mathbf{x} \qquad \left[\text{Because } \mathbf{Q} \text{ is orthogonal therefore } \mathbf{Q}^{-1} = \mathbf{Q}^T\right]$$

The new variables $(y_1 \text{ and } y_2)$ in vector **y** are obtained from the old variables $(x_1 \text{ and } x_2)$

in vector **x** by applying the matrix $\mathbf{Q}^T$ to vector **x**, that is $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$.

Substituting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ into $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$ gives

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \left[\text{Because } \mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

From this we have the new variables $y_1 = \frac{1}{\sqrt{5}}(x_1 + 2x_2)$ and $y_2 = \frac{1}{\sqrt{5}}(2x_1 - x_2)$ in terms of the old variables, x_1 and x_2.

Evaluating $f(\mathbf{y})$ by using the given result we have

$$f(\mathbf{y}) = 2y_1^2 + 12y_2^2$$

$$= 2\left[\frac{1}{\sqrt{5}}(x_1 + 2x_2)\right]^2 + 12\left[\frac{1}{\sqrt{5}}(2x_1 - x_2)\right]^2 \qquad [\text{Substituting } y_1 \text{ and } y_2]$$

$$= \frac{2}{5}\left(x_1^2 + 4x_1x_2 + 4x_2^2\right) + \frac{12}{5}\left(4x_1^2 - 4x_1x_2 + x_2^2\right) \qquad [\text{Expanding}]$$

$$= \frac{2}{5}\left[x_1^2 + 4x_1x_2 + 4x_2^2 + 6\left(4x_1^2 - 4x_1x_2 + x_2^2\right)\right] \qquad \left[\text{Taking out } \frac{2}{5} \text{ because } \frac{12}{5} = 6\left(\frac{2}{5}\right)\right]$$

$$= \frac{2}{5}\left[25x_1^2 - 20x_1x_2 + 10x_2^2\right]$$

$$\underbrace{=}_{\text{Taking out 5}} 2\left[5x_1^2 - 4x_1x_2 + 2x_2^2\right] = 10x_1^2 - 8x_1x_2 + 4x_2^2$$

We have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 10x_1^2 - 8x_1x_2 + 4x_2^2 = 2y_1^2 + 12y_2^2 = \mathbf{y}^T\mathbf{D}\mathbf{y}$$

We have $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$ which means we have converted a given quadratic into diagonal form which is easier to deal with because we only have sum of squares $2y_1^2 + 12y_2^2$ and no cross products.

Proposition (7.38). Let $\mathbf{x} = \mathbf{Q}\mathbf{y}$ where **Q** is an orthogonal matrix which diagonalizes a symmetric matrix **A**, and **D** be the diagonal eigenvalue matrix. Then any given quadratic form can be written in diagonal form as:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$$

Proof.

Consider the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{Q}\mathbf{y})^T\mathbf{A}(\mathbf{Q}\mathbf{y}) \qquad [\text{Substituting } \mathbf{x} = \mathbf{Q}\mathbf{y}]$$

$$= \left(\mathbf{y}^T\mathbf{Q}^T\right)\mathbf{A}(\mathbf{Q}\mathbf{y}) \qquad \left[\text{Using } (\mathbf{X}\mathbf{Y})^T = \mathbf{Y}^T\mathbf{X}^T\right]$$

$$= \mathbf{y}^T\left(\mathbf{Q}^T\mathbf{A}\mathbf{Q}\right)\mathbf{y}$$

$$= \mathbf{y}^T\mathbf{D}\mathbf{y} \qquad \left[\text{By (7.24) we have } \mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D}\right]$$

■

Actually we can state and prove the following more useful result called ‘The Principle Axes Theorem’.

The Principle Axes Theorem (7.39). A quadratic form in n variables $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is given by

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1,\ \lambda_2,\ \cdots$ and λ_n are the eigenvalues of the matrix $\mathbf{A}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Proof.

By the above Proposition (7.38) we have $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$ where $\mathbf{D}$ is a diagonal matrix with the eigenvalues λ of matrix $\mathbf{A}$ as entries on the leading diagonal.

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= \mathbf{y}^T\mathbf{D}\mathbf{y} \\
&= \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\
&= \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}\begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{pmatrix} \\
&= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2
\end{aligned}$$

■

Remember the advantage of writing

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

is that there are *no* cross product terms such as $y_1y_2,\ y_1y_3,\ y_2y_3, \ldots$.

In a nutshell the Principle Axes Theorem says we can write any quadratic form as sum of squares with the coefficients given by the eigenvalues.

Example 46

Transform $-x^2 + 2xy - y^2 - z^2$ into diagonal form.

<u>Solution</u>

First we need to find the matrix $\mathbf{A}$ such that $-x^2 + 2xy - y^2 - z^2 = \mathbf{x}^T\mathbf{A}\mathbf{x}$.

The entries on the leading diagonal are the coefficients of $x^2,\ y^2$ and z^2 which are -1 in each case. The only other non-zero entry is half the xy coefficient (2) which is 1. Thus

$$-x^2 + 2xy - y^2 - z^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} \text{ where } \mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

What do we need to find so that the quadratic form has no cross-product terms?

The eigenvalues of the symmetric matrix $\mathbf{A}$.

Verify that the eigenvalues of matrix $\mathbf{A}$ are $\lambda_1 = 0,\ \lambda_2 = -1$ and $\lambda_3 = -2$.

Let the new variables be $x',\ y'$ and z'. Therefore using the above Principle Axes Theorem (7.39) we have

$$\begin{aligned}
-x^2 + 2xy - y^2 - z^2 &= \lambda_1 (x')^2 + \lambda_2 (y')^2 + \lambda_3 (z')^2 \\
&= 0(x')^2 + (-1)(y')^2 + (-2)(z')^2 \\
&= -(y')^2 - 2(z')^2
\end{aligned}$$

In the above example we converted:

$$-x^2 \quad + \quad 2xy \quad - \quad y^2 \quad - \quad z^2 \quad \Longrightarrow \quad -(y')^2 \quad - \quad 2(z')^2$$

It is simpler to use the sum of squares on the Right Hand Side.
Note that to write a given quadratic form *without* any cross-products we only need to find the eigenvalues of the matrix.
In order to find the relationship between $x_1,\ x_2,\ x_3$ and the new variables $y_1,\ y_2,\ y_3$ we need to find the eigenvector matrix **Q** because $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$.
If **x** is a non-zero vector then $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$ is also non-zero because **Q** is an orthogonal matrix.

F4 Definite Matrices

In this subsection we discuss positive, negative and indefinite matrices.
A symmetric matrix **A** is *positive definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is a *positive* real number for all non-zero vectors **x**. The formal definition is:

Definition (7.40). A symmetric matrix **A** is *positive definite* $\Leftrightarrow$ the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is *positive* for *all* non-zero vectors **x**. A positive definite matrix **A** is denoted by $\mathbf{A} > 0$.

The matrix $\mathbf{A} = \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}$ of Example 44 is positive definite because

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \quad = \quad \mathbf{y}^T\mathbf{D}\mathbf{y} \quad = \quad 2y_1^2 \quad + \quad 12y_2^2 \qquad (*)$$

It is easier to check a matrix is positive definite (or negative definite) in diagonal form. For example in (*) we can see straight away that the Right Hand Side is a positive real number so matrix **A** is positive definite, $\mathbf{A} > 0$.
The identity matrix **I** is also positive definite because

$$\mathbf{x}^T\mathbf{I}\mathbf{x} \quad = \quad (x_1)^2 \quad + \quad (x_2)^2 \quad + \quad \cdots \quad + \quad (x_n)^2 \quad \text{or } \mathbf{I} > 0$$

Remember this quadratic form $\mathbf{x}^T\mathbf{I}\mathbf{x}$ gives the length of the vector **x** in $\mathbb{R}^n$.

Proposition (7.41). A symmetric matrix **A** is *positive* definite $\Leftrightarrow$ *all* the eigenvalues of **A** are *positive*.

What does this statement mean?

Means we can check a given matrix is positive definite by examining the polarity of the eigenvalues. For example the matrix $\mathbf{A} = \begin{pmatrix} 1 & -4 & -5 \\ 0 & 2 & -6 \\ 0 & 0 & 3 \end{pmatrix}$ is positive definite because we have a triangular matrix so the eigenvalues are the entries on the leading diagonal 1, 2 and 3 which are *all* positive. Hence $\mathbf{A} > 0$.
How do we prove this result?
We have $\Leftrightarrow$ in the statement therefore we need to prove the result both ways, $\Rightarrow$ and $\Leftarrow$.
Proof.
($\Leftarrow$). Let **x** be an arbitrary non-zero vector and *all* the eigenvalues a of matrix **A** be positive. Let **y** be the vector of new variables so by the Parallel Axes Theorem (7.39):

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &\underbrace{=}_{\text{By (7.39)}} \mathrm{a}_1 y_1^2 \quad + \quad \mathrm{a}_2 y_2^2 \quad + \quad \cdots \quad + \quad \mathrm{a}_n y_n^2 \\ &> 0 \qquad [\text{Because all the a's are positive}] \end{aligned}$$

Thus matrix **A** is positive definite.
($\Rightarrow$). Assume matrix **A** is positive definite then by the above definition (7.40):

(7.40). A symmetric matrix **A** is *positive definite* $\Leftrightarrow$ $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive ($\mathbf{x} \neq \mathbf{O}$).

This means for *all* non-zero vectors **x**:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 > 0 \quad \text{[Positive]}$$

Required to prove *all* the eigenvalues are positive. We prove this by contradiction.
Suppose one of the eigenvalues is negative or zero, that is $\lambda_k \leq 0$ [Negative or zero] for some k between 1 and *n*. Let $\mathbf{x} = \mathbf{v}_k = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ where $\mathbf{v}_k$ is an eigenvector belonging to the negative eigenvalue λ_k of matrix **A**. Then

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= (\mathbf{v}_k)^T \mathbf{A}\mathbf{v}_k && [\text{Substituting } \mathbf{x} = \mathbf{v}_k] \\ &= (\mathbf{v}_k)^T \lambda_k \mathbf{v}_k && [\text{Because } \mathbf{v}_k \text{ is an eigenvector; } \mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k] \\ &= \lambda_k (\mathbf{v}_k)^T \mathbf{v}_k \\ &= \lambda_k \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= \lambda_k \left[v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 \right] \end{aligned}$$

We are assuming $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ [Positive] for *all* non-zero vectors **x** therefore

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= (\mathbf{v}_k)^T \mathbf{A}\mathbf{v}_k \\ &= \lambda_k \left[v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 \right] > 0 \quad [\text{ Positive}] \end{aligned}$$

This implies $\lambda_k > 0$ [Positive]. We have a contradiction because $\lambda_k \leq 0$ and $\lambda_k > 0$. This means our supposition $\lambda_k \leq 0$ [Negative or zero] is incorrect so $\lambda_k > 0$. Hence *all* our eigenvalues are positive.

■

The identity matrix **I** is positive definite because all the eigenvalues are 1 [Positive].
Also if symmetric matrices **A** and **B** are of the same size and $\mathbf{A} - \mathbf{B}$ is positive definite, $\mathbf{A} - \mathbf{B} > 0$, then $\mathbf{A} > \mathbf{B}$. We can use this positive definiteness to compare matrices.

What do you think the term negative definite matrix means?
A is *negative* definite if $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is a *negative* real number for all non-zero vectors **x**.

Definition (7.42). A symmetric matrix **A** is *negative definite* $\Leftrightarrow$ the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is *negative* for *all* non-zero vectors **x**. We denote this by $\mathbf{A} < 0$.

The matrix **A** of the above Example 46 is *negative* definite because

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = -(y')^2 - 2(z')^2 \quad \text{[Negative]}$$

In diagonal form it is easy to notice that the quadratic is a *negative* real number.

Proposition (7.43). A symmetric matrix **A** is *negative* definite $\Leftrightarrow$ *all* the eigenvalues of **A** are *negative*.

Proof. Very similar to proof of Proposition (7.41). See Exercise 7(f).

The matrix $\mathbf{A}=\begin{pmatrix}-1 & 2\\ 0 & -3\end{pmatrix}$ is negative definite because both the eigenvalues are negative; -1 and -3 so by (7.43) we have $\mathbf{A}<0$.

Definition (7.44). A symmetric matrix **A** is *indefinite* $\Leftrightarrow$ the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ takes on *both* positive and negative values.

In general we say a quadratic form is *positive definite, negative definite or indefinite* if its matrix is *positive definite*, *negative definite or indefinite* respectively.
The following quadratic form is popular in advanced physics:

$$f(x,\ y,\ \ z,\ t)=x^2+y^2+z^2-t^2$$

This is an indefinite quadratic form because *f* can take both positive and negative values.

Example 47

Consider the quadratic form $f(\mathbf{x})=2x^2+4yx-4zx-y^2+8zy-z^2$. Decide whether this quadratic form is positive, negative or indefinite.

<u>Solution</u>

Putting the given quadratic form $f(\mathbf{x})=2x^2+4yx-4zx-y^2+8zy-z^2$ into $\mathbf{x}^T\mathbf{A}\mathbf{x}$ gives:

$$2x^2+4yx-4zx-y^2+8zy-z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}2 & 2 & -2\\ 2 & -1 & 4\\ -2 & 4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

You can verify that the matrix $\mathbf{A}=\begin{pmatrix}2 & 2 & -2\\ 2 & -1 & 4\\ -2 & 4 & -1\end{pmatrix}$ has the eigenvalues $a_1=3,\ a_2=3$ and $a_3=-6$. Applying the Principle Axes Theorem (7.39) which gives us the sum of squares:

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=a_1(x')^2+a_2(y')^2+a_3(z')^2$$

With $a_1=3,\ a_2=3$ and $a_3=-6$ we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=3(x')^2+3(y')^2-6(z')^2$$

What type is this quadratic form?

Indefinite because $3(x')^2+3(y')^2-6(z')^2$ could be either positive or negative.

Also matrix **A** is indefinite.

These positive, negative and indefinite matrices are used in optimization problems of finding maximum and minimum of a function $f(\mathbf{x})$ of more than one variable.

SUMMARY

A quadratic form can be written in matrix form as $\mathbf{x}^T\mathbf{A}\mathbf{x}$ where **A** is a real symmetric matrix.
We can convert a quadratic form into a diagonal form by $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$ where

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = a_1y_1^2 + a_2y_2^2 + \cdots + a_ny_n^2 \quad \text{[Sum of squares]}$$

Exercise **7(f)**

1. Determine $\mathbf{x}^T\mathbf{Ax}$ for the following:

(a) $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

(c) $\mathbf{A}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ (d) $\mathbf{A}=\begin{pmatrix}-1 & -4\\ -4 & -7\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$

(e) $\mathbf{A}=\begin{pmatrix}1/2 & -2\\ -2 & 1/3\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ (f) $\mathbf{A}=\begin{pmatrix}-1 & \text{®}\\ \text{®} & e\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$

2. Find $\mathbf{x}^T\mathbf{Ax}$ for the following:

(a) $\mathbf{A}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}2 & 2 & 2\\ 2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

(c) $\mathbf{A}=\begin{pmatrix}-1 & -2 & -7\\ -2 & -5 & -8\\ -7 & -8 & -9\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ (d) $\mathbf{A}=\begin{pmatrix}0 & 5 & 7\\ 5 & 0 & -3\\ 7 & -3 & 0\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

3. For the following quadratic forms determine the symmetric matrix **A** and write these quadratic forms as $\mathbf{x}^T\mathbf{Ax}$.

(a) $9x^2+2xy+y^2$ (b) x^2+y^2 (c) x^2-y^2 (d) $31x^2-41xy+69y^2$

(e) x^2+xy (f) $-x^2$ (g) xy (h) xy^2

4. Express the following quadratic forms into $\mathbf{x}^T\mathbf{Ax}$:

(a) $x^2+2xy+4xz+2y^2+10yz+5z^2$ (b) $-2x^2+6xy-32y^2-24yz+z^2$

(c) $-3x^2-6xy-12xz-7y^2-18yz-z^2$ (d) $3x^2+4xz+7y^2-5z^2$

(e) $-x^2-y^2-z^2$ (f) $2x^2-xy-xz$

(g) $yz+xz$ (h) $3x^2+5xy-7xz-3yz+2z^2$

5. Transform the following quadratic forms into diagonal form:

(a) $-2x^2+4xy+y^2$ (b) $2x^2-2xy+2y^2$ (c) $5x^2+24xy-5y^2$ (d) x^2+y^2

(e) $-2xy$ (f) x^2+2xy

6. Transform the following into diagonal form:

(a) $3x^2+8xy-3y^2+5z^2$ (b) $2x^2+4xy+4xz+2y^2+4yz+2z^2$

(c) $2x^2+4xy-4xz-y^2+8yz-z^2$ (d) $-x^2-y^2-z^2$ (e) $-x^2-2y^2+2yz-2z^2$

(f) $-2xy+xz$

A quadratic form $\mathbf{x}^T\mathbf{Ax}$ is called **positive semi-definite** $\Leftrightarrow$ $\mathbf{x}^T\mathbf{Ax}$ is positive or zero for all **x**. (ª $=0$ must be an eigenvalue of matrix **A** for semi-definite.)

A quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is called ***negative* semi-definite** $\Leftrightarrow$ $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is negative or zero for all $\mathbf{x}$.

7. Determine which of the quadratic forms of question 5 and 6 are positive definite, negative definite, indefinite, positive semi-definite or negative semi-definite.

8. Determine which of the following matrices are positive definite, negative definite, indefinite, positive semi-definite or negative semi-definite:

(a) $\mathbf{A}=\begin{pmatrix}1 & 4\\ 4 & 0\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}1 & 4\\ 4 & 16\end{pmatrix}$ (c) $\mathbf{A}=\begin{pmatrix}1 & 0 & 0\\ -3 & 2 & 0\\ 3 & -5 & 3\end{pmatrix}$ (d) $\mathbf{A}=\begin{pmatrix}-25 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$

The norm $\|\mathbf{A}\|$ of a symmetric matrix $\mathbf{A}$ is given by the absolute maximum eigenvalue of $\mathbf{A}$. For non-symmetric matrices $\mathbf{A}$ the norm is given by the absolute singular value of the matrix $\mathbf{A}$.

Find the matrix norm $\|\mathbf{A}\|$ in each the above cases.

[Use Matlab to find the norm of part (c).]

9. Prove Proposition (7.43).

10. Show that the symmetric matrix $\mathbf{A}=\begin{pmatrix}a & b\\ b & a\end{pmatrix}$ is positive definite $\Leftrightarrow$ $a>b>0$.

11. Let $\mathbf{I}$ be a n by n identity matrix and $\mathbf{x}$ be n by 1 column vector. Show that the quadratic form $\mathbf{x}^T\mathbf{I}\mathbf{x}$ satisfies the following:

(i) $\mathbf{x}^T\mathbf{I}\mathbf{x}=\mathbf{x}^T\mathbf{x}=\|\mathbf{x}\|^2$ (ii) $\mathbf{x}^T\mathbf{I}\mathbf{x}$ is positive definite

12. Let $\mathbf{A}$ be a positive definite matrix. Prove that

(a) $\mathbf{A}^m$ where $m\in\mathbb{N}$ is positive definite (b) $\mathbf{A}^{-1}$ is positive definite

13. Let $\mathbf{A}$ be a *negative* definite matrix. Prove that $\mathbf{A}^2$ is *positive* definite matrix.

14. Let $\mathbf{A}$ be a *negative* definite matrix of size n by n. Prove that

(a) $\det(\mathbf{A})<0$ provided n is odd (b) $tr(\mathbf{A})<0$ where tr is the trace of the matrix

15. Prove that if $\mathbf{A}$ is positive definite matrix then $\mathbf{A}^T$ is also a positive definite matrix.

16. Prove that if $\mathbf{A}$ is an invertible matrix then $\mathbf{A}^T\mathbf{A}$ is a positive definite matrix. [Hint: $\mathbf{v}^T\mathbf{v}=\|\mathbf{v}\|^2$].

17. (i) If $\mathbf{x}=\mathbf{Q}\mathbf{y}$ where $\mathbf{Q}$ is an orthogonal matrix then show that $\|\mathbf{y}\|=\|\mathbf{x}\|$.

(ii) Let $f(\mathbf{x})=\mathbf{x}^T\mathbf{A}\mathbf{x}$ be a quadratic form such that the eigenvalues of $\mathbf{A}$ satisfy $\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_n$. Prove that if $\|\mathbf{x}\|=1$ then $\lambda_1\geq f(\mathbf{x})\geq\lambda_n$.

18. Prove that a symmetric matrix which is positive or negative definite is invertible.

Brief Solutions to Exercise 7(f)

1. (a) x^2+y^2 (b) $x_1^2+x_2^2$ (c) $x^2+4xy+y^2$ (d) $-x^2-8xy-7y^2$

(e) $\frac{1}{2}x^2-4xy+\frac{1}{3}y^2$ (f) $-x^2+2\text{®}\,xy+ey^2$

2. (a) $x^2+y^2+z^2$ (b) $2x^2+4xy+4xz+2y^2+4zy+2z^2$

(c) $-x^2-4xy-14xz-5y^2-16zy-9z^2$ (d) $10xy+14xz-6zy$

3. (a) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (b) $\begin{pmatrix} x & y \end{pmatrix}\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix}$ (c) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(d) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 31 & -41/2 \\ -41/2 & 69 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (e) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(f) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (g) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (h) Not in quadratic form.

4. (a) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (b) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -2 & 3 & 0 \\ 3 & -32 & -12 \\ 0 & -12 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(c) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -3 & -3 & -6 \\ -3 & -7 & -9 \\ -6 & -9 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (d) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 0 & 7 & 0 \\ 2 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(e) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (f) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 2 & -1/2 & -1/2 \\ -1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(g) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (h) $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 3 & 5/2 & -7/2 \\ 5/2 & 0 & -3/2 \\ -7/2 & -3/2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

5. (a) $-3(x')^2+2(y')^2$ (b) $(x')^2+3(y')^2$ (c) $13(x')^2-13(y')^2$

(d) x^2+y^2 (e) $-(x')^2+(y')^2$ (f) $\dfrac{(1+\sqrt{5})(x')^2+(1-\sqrt{5})(y')^2}{2}$

6. (a) $-5(x')^2+5(y')^2+5(z')^2$ (b) $6(z')^2$ (c) $-6(x')^2+3(y')^2+3(z')^2$

(d) $-x^2-y^2-z^2$ (e) $\frac{\sqrt{5}}{2}(y')^2-\frac{\sqrt{5}}{2}(z')^2$

7. For question 5 we have (a) indefinite (b) positive definite (c) indefinite
(d) positive definite (e) indefinite (f) indefinite
For question 6 we have (a) indefinite (b) positive semi-definite (c) indefinite
(d) negative definite (e) indefinite

8. (a) Indefinite, $\|\mathbf{A}\|=4.5311$ (b) Positive semi-definite, $\|\mathbf{A}\|=17$

(c) Cannot find definiteness, $\|\mathbf{A}\|=7.25$ (d) Indefinite, $\|\mathbf{A}\|=25$

9. Very similar to the proof of Proposition (7.41).

10. Show that the characteristic equation is given by $\lambda^2 - 2a\lambda + a^2 - b^2 = 0$ then by using the quadratic formula derive $\lambda = a \pm b$.

11. (i) Show that $\mathbf{x}^T \mathbf{I}\mathbf{x} = x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} + \cdots + x_n^{\ 2} = \|\mathbf{x}\|^2$.

(ii) Eigenvalues are equal to 1 therefore positive definite.

12. Use Proposition (7.8) (a) which says that the eigenvalues of $\mathbf{A}^m$ are λ^m and (7.8) (b) which says that the eigenvalues of $\mathbf{A}^{-1}$ are λ^{-1}.

13. Apply Proposition (7.8) (a) again with λ being the eigenvalue of **A** then λ^2 is the eigenvalue of $\mathbf{A}^2$.

14. Use Proposition (7.9) (a), (b) which is $\det(\mathbf{A}) = \lambda_1 \times \cdots \times \lambda_n$ and $tr(\mathbf{A}) = \lambda_1 + \cdots + \lambda_n$.

15. By question 16 of Exercise 7B the eigenvalues of **A** and $\mathbf{A}^T$ are identical.

16. Show that $\mathbf{x}^T \left(\mathbf{A}^T \mathbf{A}\right)\mathbf{x} = (\mathbf{A}\mathbf{x})^T (\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 \geq 0$.

17. (i) Show the following $\|\mathbf{y}\|^2 = \mathbf{y}^T \mathbf{y} = \left(\mathbf{Q}^T \mathbf{x}\right)^T \mathbf{Q}^T \mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2$.

(ii) Use result of part (i) and $\|\mathbf{y}\|^2 = y_1^{\ 2} + y_2^{\ 2} + y_3^{\ 2} + \cdots + y_n^{\ 2} = 1$ also write $f(\mathbf{x})$ in diagonal form:

$$f(\mathbf{x}) = \lambda_1 y_1^{\ 2} + \lambda_2 y_2^{\ 2} + \lambda_3 y_3^{\ 2} + \cdots + \lambda_n y_n^{\ 2}$$

Complete Solutions **7(f)**

1. (a) Substituting $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ and $\mathbf{x}^T=\begin{pmatrix}x & y\end{pmatrix}$ into $\mathbf{x}^T\mathbf{A}\mathbf{x}$ gives

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\left(x^2+y^2\right)=x^2+y^2$$

(b) Identical to part (a) with *x* and *y* replaced by x_1 and x_2. Thus we have $x_1^{\,2}+x_2^{\,2}$.

(c) Substituting $\mathbf{A}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ and $\mathbf{x}^T=\begin{pmatrix}x & y\end{pmatrix}$ into $\mathbf{x}^T\mathbf{A}\mathbf{x}$ gives

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}x+2y & 2x+y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\left([x+2y]x+[2x+y]y\right)$$

$$=x^2+2xy+2xy+y^2=x^2+4xy+y^2$$

(d) Clearly these are all very similar. Substituting $\mathbf{A}=\begin{pmatrix}-1 & -4\\ -4 & -7\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ and $\mathbf{x}^T=\begin{pmatrix}x & y\end{pmatrix}$ into $\mathbf{x}^T\mathbf{A}\mathbf{x}$ gives

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}-1 & -4\\ -4 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}-x-4y & -4x-7y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\left([-x-4y]x+[-4x-7y]y\right)$$

$$=-x^2-4xy-4xy-7y^2=-x^2-8xy-7y^2$$

(e) Similarly we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1/2 & -2\\ -2 & 1/3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2}x-2y & -2x+\frac{1}{3}y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\left(\left[\frac{1}{2}x-2y\right]x+\left[-2x+\frac{1}{3}y\right]y\right)$$

$$=\frac{1}{2}x^2-2xy-2xy+\frac{1}{3}y^2=\frac{1}{2}x^2-4xy+\frac{1}{3}y^2$$

(f) We have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -1 & ® \\ ® & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} -x+®y & ®x+ey \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \left(\left[-x+®y\right]x+\left[®x+ey\right]y\right)$$

$$= -x^2+®xy+®xy+ey^2 = -x^2+2®xy+ey^2$$

2. (a) This time we have a matrix of size 3 by 3.

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2+y^2+z^2$$

(b) This time we have 3 by 3 matrix **A** with all the entries being 2:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 2x+2y+2z & 2x+2y+2z & 2x+2y+2z \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(\left[2x+2y+2z\right]x+\left[2x+2y+2z\right]y+\left[2x+2y+2z\right]z\right)$$

$$= 2x^2+2xy+2xz+2xy+2y^2+2zy+2xz+2yz+2z^2$$

$$= 2x^2+4xy+4xz+2y^2+4zy+2z^2$$

(c) In this case we can take out the minus sign and then evaluate $\mathbf{x}^T\mathbf{A}\mathbf{x}$:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -1 & -2 & -7 \\ -2 & -5 & -8 \\ -7 & -8 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= -\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 2 & 7 \\ 2 & 5 & 8 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= -\begin{pmatrix} x+2y+7z & 2x+5y+8z & 7x+8y+9z \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= -\left(\left[x+2y+7z\right]x + \left[2x+5y+8z\right]y + \left[7x+8y+9z\right]z\right)$$
$$= -\left(x^2+2xy+7xz+2xy+5y^2+8zy+7xz+8yz+9z^2\right)$$
$$= -\left(x^2+4xy+14xz+5y^2+16zy+9z^2\right)$$
$$= -x^2-4xy-14xz-5y^2-16zy-9z^2$$

(d) Similar to the first three we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 0 & 5 & 7 \\ 5 & 0 & -3 \\ 7 & -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 0x+5y+7z & 5x+0y-3z & 7x-3y+0z \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \left(\left[5y+7z\right]x + \left[5x-3z\right]y + \left[7x-3y\right]z\right)$$
$$= 5xy+7xz+5xy-3zy+7xz-3yz$$
$$= 10xy+14xz-6zy$$

3. In each case we need to find the symmetric matrix **A**.

(a) We are given $9x^2+2xy+y^2$ and we need to write this as $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *How?*

First we determine the symmetric matrix **A**. *What is this matrix **A** equal to?*

Well the entries on the leading diagonal are 9 and 1 because these are the coefficients of x^2 and y^2 respectively. *What are the other entries equal to?*

1 because we take half the *xy* coefficient which is 2. We have $\mathbf{A} = \begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}$.

Thus writing the given quadratic form $9x^2+2xy+y^2$ as $\mathbf{x}^T\mathbf{A}\mathbf{x}$ gives

$$9x^2+2xy+y^2 = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) We need to find the symmetric matrix **A** such that the given quadratic form x^2+y^2 can be written as $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *What are the entries of the matrix **A**?*

Since we are given x^2+y^2 therefore the leading diagonal entries are 1, 1 and the remaining entries are 0. Thus we have

$$x^2 + y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \mathbf{I} \begin{pmatrix} x \\ y \end{pmatrix}$$

In this case the symmetric matrix **A** is the identity matrix **I**.

(c) This is very similar to part (b) because we are given $x^2 - y^2$ which means that the symmetric matrix **A** has the leading diagonal entries 1, -1 and the remaining entries 0:

$$x^2 - y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(d) We are given the quadratic form $31x^2 - 41xy + 69y^2$ and we need to find the symmetric matrix **A** such that $31x^2 - 41xy + 69y^2 = \mathbf{x}^T \mathbf{A}\mathbf{x}$. *What are the entries of the matrix **A** equal to?*

The leading diagonal entries are 31 and 69 respectively. The remaining entries are half the xy coefficient which is -41 therefore $-\dfrac{41}{2}$:

$$31x^2 - 41xy + 69y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 31 & -41/2 \\ -41/2 & 69 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(e) We are given the quadratic form $x^2 + xy$. *What are the entries of the symmetric matrix **A** equal to?*

The leading diagonal entries are 1 and 0 because there is **no** y^2 term. The remaining entries are half the xy coefficient (1) which means they are ½. Thus we have

$$x^2 + xy = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(f) We need to write $-x^2$ in $\mathbf{x}^T \mathbf{A}\mathbf{x}$ form. *What are the entries of the matrix **A** equal to?*

The leading diagonal entries are -1 and 0 because there is **no** y^2 term. Since there is **no** xy coefficient therefore the remaining entries are 0. Thus we have

$$-x^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(g) *How do we write the quadratic form xy as $\mathbf{x}^T \mathbf{A}\mathbf{x}$?*

Since there are **no** x^2 and y^2 terms therefore both the leading diagonal entries are 0 and remaining entries are half the xy coefficient (1) which gives ½.

$$xy = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(h) *What do you notice about xy^2?*

It is **not** a quadratic form because xy^2 is a cubic so we **cannot** write this in $\mathbf{x}^T \mathbf{A}\mathbf{x}$ form.

4. (a) We are given the quadratic form $x^2 + 2xy + 4xz + 2y^2 + 10yz + 5z^2$ which has the three variables *x*, *y* and *z*. *What is the size of the matrix **A** in $\mathbf{x}^T \mathbf{A}\mathbf{x}$?*

We have three variables therefore we have a 3 by 3 matrix. *What are the entries of this matrix **A** equal to?*

The leading diagonal entries are the coefficients of x^2, y^2 and z^2 which are 1, 2 and 5 respectively. *What are the other entries in the matrix **A** equal to?*

Half the *xy* coefficient (2) which gives 1, half the *xz* coefficient (4) which gives 2 and half the *yz* coefficient (10) which gives 5. Thus we have

$$x^2+2xy+4xz+2y^2+10yz+5z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 5\\ 2 & 5 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

(b) We are given the quadratic form $-2x^2+6xy-32y^2-24yz+z^2$ which we need to convert into the form $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *What is the matrix* ***A*** *equal to?*

The leading diagonal entries are the coefficients of x^2, y^2 and z^2 which are -2, -32 and 1 respectively. *What are the other entries in the matrix* ***A*** *equal to?*

Half the *xy* coefficient (6) which gives 3 and half the *yz* coefficient (-24) which gives -12. Note that there is **no** *xz* coefficient which means this entry is 0. Thus we have

$$-2x^2+6xy-32y^2-24yz+z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}-2 & 3 & 0\\ 3 & -32 & -12\\ 0 & -12 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

(c) We have $-3x^2-6xy-12xz-7y^2-18yz-z^2$ and we need to convert this into $\mathbf{x}^T\mathbf{A}\mathbf{x}$ form. *How?*

We find the symmetric matrix **A**. The leading diagonal entries are -3, -7 and -1. The remaining entries are half the cross-product terms. We have

$$-3x^2-6xy-12xz-7y^2-18yz-z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}-3 & -3 & -6\\ -3 & -7 & -9\\ -6 & -9 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

(d) We are given the quadratic form $3x^2+4xz+7y^2-5z^2$ and we need to convert this into $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *What is the symmetric matrix* ***A*** *equal to?*

The leading diagonal entries are 3, 7 and -5. The only other non-zero entry in the matrix is half the xz coefficient (4) which gives 2. We have

$$3x^2+4xz+7y^2-5z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}3 & 0 & 2\\ 0 & 7 & 0\\ 2 & 0 & -5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

(e) For the quadratic form $-x^2-y^2-z^2$ the symmetric matrix **A** only has non-zero entries on the leading diagonal. *Why?*

Because there **no** cross-product terms such as *xy*, *xz* and *yz*. *What are the leading diagonal entries?*

All are -1. We have

$$-x^2-y^2-z^2=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

(f) We are given the quadratic form $2x^2-xy-xz$. *What are the entries of the symmetric matrix* ***A*** *equal to?*

The only non-zero entry on the leading diagonal is 2 (which is the first entry on the diagonal) because we have **no** y^2 or z^2. The other entries are given by finding half the *xy*, *xz* and *yz* coefficients which are $-1/2$, $-1/2$ and 0 respectively. We have

$$2x^2 - xy - xz = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & -1/2 & -1/2 \\ -1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(g) We can write $yz + xz = xz + yz$. *What are the entries of the symmetric matrix **A** equal to?*

We have a 3 by 3 matrix because we have 3 variables $x,\ y,\ z$. The leading diagonal entries are **all** zero because we have **no** $x^2,\ y^2$ or z^2. The remaining entries are half *xy*, *xz* and *yz* coefficients. These are 0, ½ and ½. We have

$$yz + xz = xz + yz = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(h) *What is the symmetric matrix **A** for the given quadratic form*

$$3x^2 + 5xy - 7xz - 3yz + 2z^2\ ?$$

The leading diagonal entries are 3, 0 and 2 respectively. The remaining entries are half *xy*, *xz* and *yz* coefficients. These are $5/2$, $-7/2$ and $-3/2$ respectively. We have

$$3x^2 + 5xy - 7xz - 3yz + 2z^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & 5/2 & -7/2 \\ 5/2 & 0 & -3/2 \\ -7/2 & -3/2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

5. (a) We are given the quadratic form $-2x^2 + 4xy + y^2$. First we find the symmetric matrix **A** which represents this quadratic form in $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *What is **A** equal to?*

The leading diagonal entries are -2 and 1. The remaining entry is half the *xy* coefficient (4) which is 2. The symmetric matrix $\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$ and

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

How do we write the given quadratic form $-2x^2 + 4xy + y^2$ into one without any cross-product terms such as xy?

We apply the Principles Axes Theorem (7.39) which says

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \lambda_1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \cdots + \lambda_n y_n^{\,2}$$

where λ 's are the eigenvalues of the matrix **A**. In this case we transform from the variables x and y to x' and y' respectively. Applying the above theorem (7.39) we have

$$-2x^2 + 4xy + y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} = \lambda_1 (x')^2 + \lambda_2 (y')^2$$

where λ_1 and λ_2 are the eigenvalues of the matrix **A**.

*What are the eigenvalues of matrix **A**?*

You can verify in your own time that $\lambda_1 = -3$ and $\lambda_2 = 2$. Therefore substituting these $\lambda_1 = -3$ and $\lambda_2 = 2$ into $-2x^2 + 4xy + y^2 = \lambda_1 (x')^2 + \lambda_2 (y')^2$ gives

$$\begin{aligned} -2x^2 + 4xy + y^2 &= (-3)(x')^2 + 2(y')^2 \\ &= -3(x')^2 + 2(y')^2 \end{aligned}$$

(b) We are given the quadratic form $2x^2 - 2xy + 2y^2$. First we find the symmetric matrix **A** which represents this quadratic form in $\mathbf{x}^T\mathbf{A}\mathbf{x}$. *What is **A** equal to?*
The leading diagonal entries are 2 and 2. The remaining entry is half the *xy* coefficient (-2) which is -1. The symmetric matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and

$$2x^2 - 2xy + 2y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

How do we write the given quadratic form $2x^2 - 2xy + 2y^2$ into one without any cross-product terms such as xy?
We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x* and *y* to x' and y' respectively. Applying (7.39) we have

$$2x^2 - 2xy + 2y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} = a_1(x')^2 + a_2(y')^2$$

*What are the eigenvalues of matrix **A**?*
You can verify in your own time that $a_1 = 1$ and $a_2 = 3$. Therefore substituting these $a_1 = 1$ and $a_2 = 3$ into $2x^2 - 2xy + 2y^2 = a_1(x')^2 + a_2(y')^2$ gives

$$\begin{aligned} 2x^2 - 2xy + 2y^2 &= 1(x')^2 + 3(y')^2 \\ &= (x')^2 + 3(y')^2 \end{aligned}$$

(c) We are given the quadratic form $5x^2 + 24xy - 5y^2$. *What is the symmetric matrix **A** equal to?*
The leading diagonal entries are 5 and -5. The remaining entry is half the *xy* coefficient (24) which is 12. The symmetric matrix $\mathbf{A} = \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$ and

$$5x^2 + 24xy - 5y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

How do we write the given quadratic form $5x^2 + 24xy - 5y^2$ into one without any cross-product terms such as xy?
We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x* and *y* to x' and y' respectively. Applying (7.39) we have

$$5x^2 + 24xy - 5y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} = a_1(x')^2 + a_2(y')^2$$

*What are the eigenvalues of matrix **A**?*
You can verify in your own time that $a_1 = 13$ and $a_2 = -13$. Therefore substituting these $a_1 = 13$ and $a_2 = -13$ into $5x^2 + 24xy - 5y^2 = a_1(x')^2 + a_2(y')^2$ gives

$$5x^2 + 24xy - 5y^2 = 13(x')^2 - 13(y')^2$$

(d) We are given the quadratic form $x^2 + y^2$. *What do you notice about this $x^2 + y^2$?*
It does **not** contain any cross-product terms which means it is already in diagonal form, thus $x^2 + y^2 = x^2 + y^2$.
(e) We are given the quadratic form $-2xy$. *What is the symmetric matrix **A** equal to?*

The leading diagonal entries are 0 and 0. The remaining entry is half the *xy* coefficient (-2) which is -1. The symmetric matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and

$$-2xy = \mathbf{x}^T \mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

How do we write the given quadratic form $-2xy$ into one ***without*** *any cross-product terms such as xy?*

We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x* and *y* to x' and y' respectively. Applying (7.39) we have

$$-2xy = \mathbf{x}^T \mathbf{A}\mathbf{x} = a_1 (x')^2 + a_2 (y')^2$$

*What are the eigenvalues of matrix **A**?*

You can verify in your own time that $a_1 = -1$ and $a_2 = 1$. Therefore substituting these $a_1 = -1$ and $a_2 = 1$ into $-2xy = a_1 (x')^2 + a_2 (y')^2$ gives

$$-2xy = -1(x')^2 + 1(y')^2 = -(x')^2 + (y')^2$$

(f) We are given the quadratic form $x^2 + 2xy$. *What is the symmetric matrix **A** equal to?*

The leading diagonal entries are 1 and 0. The remaining entry is half the *xy* coefficient (2) which is 1. The symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (this matrix is also known as the Fibonacci matrix) and

$$x^2 + 2xy = \mathbf{x}^T \mathbf{A}\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

How do we write the given quadratic form $x^2 + 2xy$ into one ***without*** *any cross-product terms such as xy?*

We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x* and *y* to x' and y' respectively. Applying (7.39) we have

$$x^2 + 2xy = \mathbf{x}^T \mathbf{A}\mathbf{x} = a_1 (x')^2 + a_2 (y')^2$$

*What are the eigenvalues of matrix **A**?*

You can verify in your own time that $a_1 = \frac{1+\sqrt{5}}{2}$ and $a_2 = \frac{1-\sqrt{5}}{2}$. Therefore substituting these $a_1 = \frac{1+\sqrt{5}}{2}$ and $a_2 = \frac{1-\sqrt{5}}{2}$ into $x^2 + 2xy = a_1 (x')^2 + a_2 (y')^2$ gives

$$\begin{aligned} x^2 + 2xy &= \left(\frac{1+\sqrt{5}}{2}\right)(x')^2 + \left(\frac{1-\sqrt{5}}{2}\right)(y')^2 \\ &= \frac{\left(1+\sqrt{5}\right)(x')^2 + \left(1-\sqrt{5}\right)(y')^2}{2} \end{aligned}$$

6. (a) In this case we are given the quadratic form $3x^2 + 8xy - 3y^2 + 5z^2$ which means it is in three variables *x*, *y* and *z*. Hence we need a 3 by 3 matrix **A**. *What is the symmetric matrix **A** equal to?*

The leading diagonal entries are 3, -3 and 5. The only other non-zero entry in the matrix is half the *xy* coefficient (8) which is 4. Thus we have

$$3x^2+8xy-3y^2+5z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}3 & 4 & 0\\ 4 & -3 & 0\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

We need to write the given quadratic form $3x^2+8xy-3y^2+5z^2$ without any cross-product terms such as *xy*, *xz* and *yz*. *How?*
We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x*, *y* and *z* to x', y' and z' respectively. Applying (7.39) we have

$$3x^2+8xy-3y^2+5z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=\alpha_1\left(x'\right)^2+\alpha_2\left(y'\right)^2+\alpha_3\left(z'\right)^2$$

where α_1, α_2 and α_3 are the eigenvalues of the matrix **A**. *What are the eigenvalues of **A**?*
You can verify in your own time that $\alpha_1=-5,\ \alpha_2=5$ and $\alpha_3=5$. Substituting these into the above gives

$$3x^2+8xy-3y^2+5z^2=-5\left(x'\right)^2+5\left(y'\right)^2+5\left(z'\right)^2$$

(b) We are given the quadratic form $2x^2+4xy+4xz+2y^2+4yz+2z^2$. *What is the symmetric matrix **A** equal to?*
The leading diagonal entries are 2, 2 and 2. The remaining entries in the matrix are half the *xy*, *xz* and *yz* coefficients which are all 4 therefore the matrix entries are 2. Thus we have

$$2x^2+4xy+4xz+2y^2+4yz+2z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}2 & 2 & 2\\ 2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

We need to write the given quadratic form $2x^2+4xy+4xz+2y^2+4yz+2z^2$ without any cross-product terms such as *xy*, *xz* and *yz*. *How?*
We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x*, *y* and *z* to x', y' and z' respectively. Applying (7.39) we have

$$2x^2+4xy+4xz+2y^2+4yz+2z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=\alpha_1\left(x'\right)^2+\alpha_2\left(y'\right)^2+\alpha_3\left(z'\right)^2$$

where α_1, α_2 and α_3 are the eigenvalues of the matrix **A**. *What are the eigenvalues of **A**?*
You can verify in your own time that $\alpha_1=0,\ \alpha_2=0$ and $\alpha_3=6$. Substituting these into the above gives

$$\begin{aligned}2x^2+4xy+4xz+2y^2+4yz+2z^2&=0\left(x'\right)^2+0\left(y'\right)^2+6\left(z'\right)^2\\&=6\left(z'\right)^2\end{aligned}$$

(c) We are given the quadratic form $2x^2+4xy-4xz-y^2+8yz-z^2$. *What is the symmetric matrix **A** equal to?*
The leading diagonal entries are 2, -1 and -1. The remaining entries in the matrix are half the *xy*, *xz* and *yz* coefficients which are 4, -4 and 8 respectively. Therefore the entries on the non leading diagonal are 2, -2 and 4. We have

$$2x^2+4xy-4xz-y^2+8yz-z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}2 & 2 & -2\\ 2 & -1 & 4\\ -2 & 4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

We need to write the given quadratic form $2x^2+4xy-4xz-y^2+8yz-z^2$ without any cross-product terms such as *xy*, *xz* and *yz*. *How?*
We apply the Principles Axes Theorem (7.39) which in this case we transform from the variables *x*, *y* and *z* to x', y' and z' respectively. Applying (7.39) we have

$$2x^2+4xy-4xz-y^2+8yz-z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}=a_1(x')^2+a_2(y')^2+a_3(z')^2$$

where a_1, a_2 and a_3 are the eigenvalues of the matrix **A**. *What are the eigenvalues of **A**?*
You can verify in your own time that $a_1=-6,\ a_2=3$ and $a_3=3$. Substituting these into the above gives

$$2x^2+4xy-4xz-y^2+8yz-z^2=-6(x')^2+3(y')^2+3(z')^2$$

(d) *What do you notice about the given quadratic form* $-x^2-y^2-z^2$ *?*
It is already in diagonal form, that is there are **no** cross product terms which means that

$$-x^2-y^2-z^2=-x^2-y^2-z^2$$

(e) First we need to find the symmetric matrix **A** such that $-x^2-2y^2+2yz-2z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}$.
*What is the matrix **A** equal to in this case?*
The entries on the leading diagonal are the coefficients of x^2, y^2 and z^2 which are -1 -2 and -2. The only other non-zero entry is half the *yz* coefficient (2) which is 1. We have

$$-x^2-2y^2+2yz-2z^2=\mathbf{x}^T\mathbf{A}\mathbf{x}\ \text{ where }\ \mathbf{A}=\begin{pmatrix}-1&0&0\\0&-2&1\\0&1&-2\end{pmatrix}$$

We need to find the eigenvalues of matrix **A**. Verify that the eigenvalues of matrix **A** are $a_1=-3,\ a_2=-1$ and $a_3=-1$.
Let the new quadratic form be in the variables x', y' and z'. Therefore using the above Principle Axes Theorem (7.39) we have

$$\begin{aligned}-x^2-2y^2+2yz-2z^2&=a_1(x')^2+a_2(y')^2+a_3(z')^2\\&=-3(x')^2+(-1)(y')^2+(-1)(z')^2\\&=-3(x')^2-(y')^2-(z')^2\end{aligned}$$

(f) We are given the quadratic form $-2xy+xz$. *How do we convert this into diagonal form?*
First we find the symmetric matrix **A**. *What are the diagonal entries in the matrix **A** equal to?*
There are **no** x^2, y^2 or z^2 terms therefore the leading diagonal entries are **all** zero. *What are the remaining entries in the matrix **A** equal to?*
There are only two non-zero entries which are half the *xy* and *xz* coefficients (-2 and 1 respectively) which are -1 and 1/2. Therefore we have

$$-2xy+xz=\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}0&-1&1/2\\-1&0&0\\1/2&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

What are the eigenvalues of the matrix $\mathbf{A}=\begin{pmatrix}0 & -1 & 1/2\\ -1 & 0 & 0\\ 1/2 & 0 & 0\end{pmatrix}$?

You can verify that $\lambda_1=0$, $\lambda_2=\frac{\sqrt{5}}{2}$ and $\lambda_3=-\frac{\sqrt{5}}{2}$. Hence we have

$$-2xy=\mathbf{x}^T\mathbf{Ax}=\lambda_1(x')^2+\lambda_2(y')^2+\lambda_3(z')^2$$
$$=0(x')^2+\frac{\sqrt{5}}{2}(y')^2-\frac{\sqrt{5}}{2}(z')^2=\frac{\sqrt{5}}{2}(y')^2-\frac{\sqrt{5}}{2}(z')^2$$

7. For question 5 we have the following diagonal forms:
(a) $\mathbf{x}^T\mathbf{Ax}=-3(x')^2+2(y')^2$ can take on both positive and negative values therefore the quadratic form is indefinite.
(b) $\mathbf{x}^T\mathbf{Ax}=(x')^2+3(y')^2$ is a quadratic form which can only take on positive values for **all** $\mathbf{x}\neq\mathbf{O}$, therefore it is positive definite.
(c) Clearly the quadratic form $\mathbf{x}^T\mathbf{Ax}=13(x')^2-13(y')^2$ can take on both positive and negative values therefore it is indefinite quadratic form.
(d) The quadratic form $\mathbf{x}^T\mathbf{Ax}=x^2+y^2$ can only take on positive values therefore it is positive definite.
(e) $\mathbf{x}^T\mathbf{Ax}=-(x')^2+(y')^2$ is a quadratic form which can take on both positive and negative values therefore it is an indefinite quadratic form.
(f) The quadratic form $\mathbf{x}^T\mathbf{Ax}=\frac{(1+\sqrt{5})(x')^2+(1-\sqrt{5})(y')^2}{2}$ can take on both positive and negative values therefore it is indefinite.
For question 6 we have the following diagonal forms:
(a) $-5(x')^2+5(y')^2+5(z')^2$ is a quadratic form which can take on both positive and negative values therefore it is indefinite.
(b) The quadratic form $6(z')^2$ can have positive or zero values therefore it is positive semi-definite.
(c) The quadratic form $-6(x')^2+3(y')^2+3(z')^2$ can take on both positive and negative values therefore it is indefinite quadratic form.
(d) $-x^2-y^2-z^2$ can only take on negative values for **all** $\mathbf{x}\neq\mathbf{O}$ therefore the quadratic form is negative definite.
(e) The quadratic form $\frac{\sqrt{5}}{2}(y')^2-\frac{\sqrt{5}}{2}(z')^2$ can take on both positive and negative values therefore it is an indefinite quadratic form.

8. (a) The eigenvalues of $\mathbf{A}=\begin{pmatrix}1 & 4\\ 4 & 0\end{pmatrix}$ are $\lambda_1=-3.5311$, $\lambda_2=4.5311$ so the matrix **A** is indefinite.

(b) Since the bottom row is a multiple (4) of the first row in matrix $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix}$ so two rows of **A** are linearly dependent so 0 must be an eigenvalue of matrix **A**. The other eigenvalue is 17 so matrix **A** is positive semi-definite.

(c) The given matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 3 & -5 & 3 \end{pmatrix}$ is *not* a symmetric matrix so we cannot find whether it is positive, negative or indefinite matrix.

(d) The eigenvalues of the given matrix $\mathbf{A} = \begin{pmatrix} -25 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ are $-25,\ 2$ and 3 so the matrix is indefinite.

The matrix norm is the maximum absolute value:

(a) $\|\mathbf{A}\| = 4.5311$ (b) $\|\mathbf{A}\| = 17$ (d) $\|\mathbf{A}\| = 25$

(c) The given matrix is *not* symmetric. We need to find $\mathbf{A}^T\mathbf{A}$:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 19 & -21 & 9 \\ -21 & 29 & -15 \\ 9 & -15 & 9 \end{pmatrix}$$

The eigenvalues of this matrix $\mathbf{A}^T\mathbf{A}$ are $a_1 = 0.1586,\ a_2 = 4.3219$ and $a_3 = 52.5195$.
Hence

$$\|\mathbf{A}\| = \sqrt{52.5195} = 7.25$$

9. The proof is very similar to the proof of Proposition (7.41).

Proposition (7.43). A symmetric matrix **A** is **negative** definite $\Leftrightarrow$ **all** the eigenvalues of **A** are **negative**.

How do we prove this result?

Since we have an if and only if ($\Leftrightarrow$) therefore we need to prove the result both ways $\Rightarrow$ and $\Leftarrow$.

Proof.

($\Leftarrow$). Let **x** be an arbitrary non-zero vector and **all** the eigenvalues a of matrix **A** be negative. Let **y** be the vector of new variables which are given by $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$ where **Q** is an orthogonal matrix such that $\mathbf{Q}^T\mathbf{AQ} = \mathbf{D}$. Then by (7.39) we have

$$\begin{aligned} \mathbf{x}^T\mathbf{Ax} &= \mathbf{y}^T\mathbf{Dy} \\ &\underbrace{=}_{\text{By (7.31)}} a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + \cdots + a_n y_n^2 \\ &< 0 \qquad \left[\text{Because all the } a\text{'s are negative}\right] \end{aligned}$$

Thus matrix **A** is negative definite.

($\Rightarrow$). Assume matrix **A** is negative definite which means for **all** non-zero vectors **x**

$$\mathbf{x}^T\mathbf{Ax} = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + \cdots + a_n y_n^2 < 0$$

Suppose $\lambda_k \geq 0$ [Positive or zero] for some k between 1 and n. Let $\mathbf{x} = \mathbf{v}_k = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ where

$\mathbf{v}_k$ is an eigenvector belonging to the eigenvalue λ_k of the matrix **A**. Then

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= (\mathbf{v}_k)^T\,\mathbf{A}\mathbf{v}_k \\
&= (\mathbf{v}_k)^T\,\lambda_k\mathbf{v}_k \qquad \left[\text{Because } \mathbf{v}_k \text{ is e.vector } \mathbf{A}\mathbf{v}_k = \lambda_k\mathbf{v}_k\right] \\
&= \lambda_k\,(\mathbf{v}_k)^T\,\mathbf{v}_k \\
&= \lambda_k\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\
&= \lambda_k\left[v_1^{\,2} + v_2^{\,2} + v_3^{\,2} + \cdots + v_n^{\,2}\right]
\end{aligned}$$

Since $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ [Negative] for **all** non-zero vectors **x** therefore

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= (\mathbf{v}_k)^T\,\mathbf{A}\mathbf{v}_k \\
&= \lambda_k\left[v_1^{\,2} + v_2^{\,2} + v_3^{\,2} + \cdots + v_n^{\,2}\right] < 0 \qquad [\text{ Negative}]
\end{aligned}$$

This gives $\lambda_k < 0$. We have a contradiction because $\lambda_k \geq 0$ and $\lambda_k < 0$. This means our supposition $\lambda_k \geq 0$ is incorrect so $\lambda_k < 0$. Hence **all** our eigenvalues are negative.

■

10. Need to prove that $\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is positive definite if an only if $a > b > 0$.

Proof. To prove that $\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is positive definite we need to show that **all** the eigenvalues of **A** are positive. The eigenvalues λ are given by

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} \\
&= (a-\lambda)^2 - b^2 \\
&= \lambda^2 - 2a\lambda + a^2 - b^2 = 0
\end{aligned}$$

We can solve the quadratic $\lambda^2 - 2a\lambda + a^2 - b^2 = 0$ by using the formula

$$\begin{aligned}
\lambda &= \frac{2a \pm \sqrt{4a^2 - 4\left(a^2 - b^2\right)}}{2} \\
&= a \pm b
\end{aligned}$$

This gives the eigenvalues $\lambda_1 = a + b$ and $\lambda_2 = a - b$. If $a > b > 0$ then $\lambda_1 > 0$ and $\lambda_2 > 0$ which means that **all** the eigenvalues are positive.
Thus **A** is positive definite $\Leftrightarrow$ $a > b > 0$.

■

11. We need to show that the quadratic form (i) $\mathbf{x}^T\mathbf{I}\mathbf{x} = \|\mathbf{x}\|^2$ and (ii) $\mathbf{x}^T\mathbf{I}\mathbf{x}$ is positive definite.

Proof.

(i) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then we have

$$\mathbf{x}^T\mathbf{I}\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = \|\mathbf{x}\|^2$$

■

Proof of (ii).

Since all the eigenvalues of the identity matrix **I** are 1 therefore the matrix **I** is positive definite so the quadratic form $\mathbf{x}^T\mathbf{I}\mathbf{x}$ is positive definite.

■

12. We need to prove that if **A** is a positive definite matrix then

(a) $\mathbf{A}^m$ where $m \in \mathbb{N}$ is positive definite.

(b) $\mathbf{A}^{-1}$ is positive definite

Proof of (a).

Since the given matrix **A** is positive definite therefore by Proposition (7.41) **all** the eigenvalues a of **A** are positive. By Proposition (7.8) (a):

(7.8) (a) The eigenvalues of $\mathbf{A}^m$ are a^m

We have the eigenvalues of $\mathbf{A}^m$ are a^m which means that these are also positive therefore the matrix $\mathbf{A}^m$ is positive definite.

■

Proof of (b).

Since the given matrix **A** is positive definite therefore by Proposition (7.41) **all** the eigenvalues a of **A** are positive. By Proposition (7.8) (b):

(7.8) (b) The eigenvalues of $\mathbf{A}^{-1}$ are a^{-1}

We have the eigenvalues of $\mathbf{A}^{-1}$ are $\frac{1}{a}$ which means that these are also positive therefore the matrix $\mathbf{A}^{-1}$ is positive definite.

■

13. Required to prove that if **A** is a **negative** definite matrix then $\mathbf{A}^2$ is **positive** definite matrix.

Proof.

Since the given matrix **A** is negative definite therefore by Proposition (7.43) **all** the eigenvalues a of **A** are negative. The eigenvalues of $\mathbf{A}^2$ are a^2 which means **all** the eigenvalues are positive. Hence $\mathbf{A}^2$ is positive definite matrix.

■

14. We need to prove that if **A** is a **negative** definite matrix then

(a) $\det(\mathbf{A}) < 0$ (b) $tr(\mathbf{A}) < 0$ where *tr* is the trace of the matrix

Proof of (a).

Since the given matrix **A** is negative definite therefore by Proposition (7.43):

(7.43) Symmetric matrix **A** is *negative* definite $\Leftrightarrow$ *all* the eigenvalues of **A** are *negative*.

All the eigenvalues $a_1,\ a_2,\ a_3,\ \cdots$ and a_n of **A** are negative. By Proposition (7.9) (a):

(7.9) (a) $\det(\mathbf{A}) = a_1 \times a_2 \times a_3 \times \cdots \times a_n$

The determinant is given by

$$\det(\mathbf{A}) = \underbrace{a_1 \times a_2 \times a_3 \times \cdots \times a_n}_{\text{Odd } n}$$

Since n is odd therefore we have a multiple of odd negative numbers which means that it is negative so we have our result $\det(\mathbf{A}) < 0$.

■

Proof of (b).

Since the given matrix **A** is negative definite therefore by Proposition (7.43):

(7.43) Symmetric matrix **A** is *negative* definite $\Leftrightarrow$ *all* the eigenvalues of **A** are *negative*.

All the eigenvalues $a_1,\ a_2,\ a_3,\ \cdots$ and a_n of **A** are negative. By Proposition (7.9) (b):

(7.9) (b) $tr(\mathbf{A}) = a_1 + a_2 + a_3 + \cdots + a_n$

The trace of the matrix **A** is given by

$$tr(\mathbf{A}) = a_1 + a_2 + a_3 + \cdots + a_n$$

Since **all** the a 's are negative therefore $tr(\mathbf{A}) < 0$.

■

15. We need to prove that if **A** is positive definite matrix then $\mathbf{A}^T$ is also a positive definite matrix.

Proof.

Let a be the eigenvalues of the matrix **A**. Since **A** is positive definite therefore all the eigenvalues of **A** are positive. By question 16 of Exercise 7(b) we know that $\mathbf{A}^T$ has the same eigenvalues a therefore all the eigenvalues of $\mathbf{A}^T$ are positive so $\mathbf{A}^T$ is a positive definite matrix.

■

16. We need to prove that $\mathbf{A}^T\mathbf{A}$ is positive definite provided that **A** is an invertible matrix.

Proof.

First we need to show that $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A}$$

Thus $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix. Next we examine the quadratic form $\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}$:

$$\begin{aligned}\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} &= (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) \\ &= \|\mathbf{A}\mathbf{x}\|^2 \geq 0 \qquad \text{[By Hint]}\end{aligned}$$

Thus $\|\mathbf{Ax}\|^2 \geq 0$. Consider the case $\|\mathbf{Ax}\|^2 = 0$, then $\mathbf{Ax} = \mathbf{O}$ and we are given that **A** is invertible so multiplying both sides of $\mathbf{Ax} = \mathbf{O}$ by $\mathbf{A}^{-1}$ gives $\mathbf{x} = \mathbf{O}$. Hence for $\mathbf{x} \neq \mathbf{O}$ $\|\mathbf{Ax}\|^2 > 0$ which means that $\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} = \|\mathbf{Ax}\|^2 > 0$ therefore the matrix $\mathbf{A}^T\mathbf{A}$ is positive definite.

■

17. (i) We are required to prove that if $\mathbf{x} = \mathbf{Qy}$ where **Q** is an orthogonal matrix then $\|\mathbf{y}\| = \|\mathbf{x}\|$.

Proof.

Since **Q** is an orthogonal matrix therefore $\mathbf{Q}^{-1} = \mathbf{Q}^T$. We have $\|\mathbf{y}\|^2 = \mathbf{y}^T\mathbf{y}$. *What is* ***y*** *equal to?*

We are given that $\mathbf{x} = \mathbf{Qy}$ so multiplying both sides of this $\mathbf{x} = \mathbf{Qy}$ by $\mathbf{Q}^T$ (because $\mathbf{Q}^{-1} = \mathbf{Q}^T$) gives

$$\mathbf{Q}^T\mathbf{x} = \underbrace{\mathbf{Q}^T\mathbf{Q}}_{=\mathbf{I}}\mathbf{y} = \mathbf{y} \text{ or } \mathbf{y} = \mathbf{Q}^T\mathbf{x}$$

Substituting this $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$ into the above $\|\mathbf{y}\|^2 = \mathbf{y}^T\mathbf{y}$ yields

$$\begin{aligned}\|\mathbf{y}\|^2 = \mathbf{y}^T\mathbf{y} &= \left(\mathbf{Q}^T\mathbf{x}\right)^T\mathbf{Q}^T\mathbf{x} \\ &= \mathbf{x}^T\underbrace{\mathbf{Q}\mathbf{Q}^T}_{=\mathbf{I}}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{I}\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2\end{aligned}$$

We have $\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2$ and taking the square root of both sides of this gives our result $\|\mathbf{y}\| = \|\mathbf{x}\|$.

■

(ii) We need to prove if the eigenvalues of **A** satisfy $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ and $\|\mathbf{x}\| = 1$ then $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$.

Proof.

By the above part (i) we have $\|\mathbf{y}\| = \|\mathbf{x}\| = 1$. This means that $\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 = 1$ and

$$\|\mathbf{y}\|^2 = y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2 = 1 \qquad (*)$$

The quadratic form $f(\mathbf{x}) = \mathbf{x}^T\mathbf{Ax}$ in diagonal form is given by

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{Ax} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2 \qquad (**)$$

Using the given inequality $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ we have

$$\begin{aligned}\lambda_1(1) &= \lambda_1\underbrace{\left(y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2\right)}_{=1 \text{ by } (*)} \\ &= \lambda_1 y_1^2 + \lambda_1 y_2^2 + \lambda_1 y_3^2 + \cdots + \lambda_1 y_n^2 \\ &\geq \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2 && \left[\text{Because } \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n\right] \\ &= f(\mathbf{x}) && \left[\text{By } (**)\right]\end{aligned}$$

Thus we have our first inequality $\lambda_1 \geq f(\mathbf{x})$. Using (**) we have

$$\begin{aligned} f(\mathbf{x}) &= \lambda_1 y_1^{\ 2} + \lambda_2 y_2^{\ 2} + \lambda_3 y_3^{\ 2} + \cdots + \lambda_n y_n^{\ 2} \\ &\geq \lambda_n y_1^{\ 2} + \lambda_n y_2^{\ 2} + \lambda_n y_3^{\ 2} + \cdots + \lambda_n y_n^{\ 2} \qquad \left[\text{Because } \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n\right] \\ &= \lambda_n \underbrace{\left(y_1^{\ 2} + y_2^{\ 2} + y_3^{\ 2} + \cdots + y_n^{\ 2}\right)}_{=1 \text{ by } (*)} = \lambda_n \end{aligned}$$

Hence we have our second inequality, that is $f(\mathbf{x}) \geq \lambda_n$. Putting the two inequalities together we have our result $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$.

■

18. Need to prove that a symmetric matrix which is positive or negative definite is invertible.
Proof.
Without loss of generality assume a symmetric matrix **A** is positive definite. By

> Proposition (7.41). A symmetric matrix **A** is *positive* definite $\Leftrightarrow$ *all* the eigenvalues of **A** are *positive*.

All the eigenvalues of matrix **A** are positive. By

> Proposition (7.7). A square matrix **A** is invertible (has an inverse) $\Leftrightarrow$ $\lambda = 0$ is *not* an eigenvalue of the matrix **A**.

Since $\lambda = 0$ is not an eigenvalue of matrix **A** because all the eigenvalues are positive so matrix **A** is invertible.

■