Chapter 2 : Infinite Series

Section I **Power Series**

By the end of this section you will be able to

- test for absolute convergence
- understand what is meant by a power series
- determine values of x of a power series for absolute convergence, conditional convergence and divergence

In this section we look at power series. These are important because functions such as exponential, sine, cosine etc can be defined as power series. These series contain a variable x and in this section we need to determine values of x for which the power series converges. To find these values of x we apply the ratio test which involves a lot of algebraic simplification and the application of the rules of inequalities. You also need to understand previous topics such as evaluation of limits of sequences and properties of the modulus function.

I1 **Test for Absolute Convergence**

We can setup direct tests for absolute convergence. We only need to modify our existing tests. For example we can use a version of the ratio test given in section E for absolute convergence.

Ratio Test for Absolute Convergence (2.22).

Let $\sum(a_n)$ be a series in $\mathbb{R}$ where $\forall n \in \mathbb{N}, \quad a_n \neq 0$ $\left[\text{Not Zero}\right]$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \qquad \text{[Ratio of the } (n+1)\text{th term to the nth term]}$$

(I) If $L < 1$ then the series $\sum(a_n)$ converges absolutely.

(II) If $L > 1$ then the series $\sum(a_n)$ diverges.

(III) If $L = 1$ the test fails and we **cannot** conclude whether the series converges absolutely, converges conditionally or diverges.

Note: *What does this ratio test mean?*

The ratio test is divided into three parts. Part (I) says that if the limit of the absolute value of the (n+1)th term divided by the nth term of a given series is less than 1 then the series converges absolutely. *What does part (II) mean?*

It means that if the limit of absolute value of the (n+1)th term divided by the nth term of a given series is greater than 1 then the series diverges. *What does part (III) mean?*

If the limit of absolute value of (n+1)th term divided by the nth term of a given series is equal to 1 then we **cannot** say whether the series converges absolutely or conditionally or diverges. The test fails.

Proof of (I).

We apply the ratio test of section E that is (2.14) to the series $\sum|a_n|$. *What is the nth term a_n equal to?*

$$a_n = |a_n|$$

What is the $(n+1)$th term equal to?

$$a_{n+1} = |a_{n+1}|$$

Substituting these, $a_n = |a_n|$ and $a_{n+1} = |a_{n+1}|$, into

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right)$$

$$= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \quad \left[\text{Because } \frac{|x|}{|y|} = \left|\frac{x}{y}\right|\right]$$

$$= L < 1$$

By (2.14) part (I) the series $\sum |a_n|$ converges because $L < 1$. Therefore by the definition of absolute convergence (2.20) the series $\sum (a_n)$ converges absolutely.

Proof of (II).

Similar to proof of part (II) of (2.14). See Exercise 2i.

Proof of (III).

How do we prove the test fails for L=1?

We need to produce three series one which diverges another which converges conditionally and the last which converges absolutely. All three series must have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \qquad [\text{That is } L = 1 \text{ in each case}]$$

Can you think of examples of these series?

1. $\sum\left(\frac{1}{n}\right)$ diverges
2. $\sum\left(\frac{(-1)^{n+1}}{n^2}\right)$ converges absolutely
3. $\sum\left(\frac{(-1)^{n+1}}{n}\right)$ converges conditionally

You can check all of these have $L = 1$. See Exercise 2i.

We can apply this ratio test for solving the next problem. You will need to understand your work on limits of sequences because they are used to evaluate L. For example you should be familiar with

$$\lim_{n\to\infty}\left(c\, f(n)\right) = c \lim_{n\to\infty}\left(f(n)\right) \quad \text{where } c \text{ is a constant}$$

Also you need to be confident in using properties of modulus function such as

$$|xy| = |x||y|$$

Example 45

Determine the values of x for which the following series

$$\sum\left(\frac{x^n}{n}\right) \qquad (\text{where } x \in \mathbb{R})$$

(a) diverges
(b) converges absolutely
(c) converges conditionally

(2.14) (I) If $L < 1$ then the series $\sum (a_n)$ converges

(2.20) If $\sum |a_n|$ converges then $\sum (a_n)$ converges absolutely

Solution

To use the ratio test for absolute convergence we need to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

What is a_n equal to in this case?

It is the nth term so we have $a_n = \frac{x^n}{n}$.

How do we find a_{n+1}?

a_{n+1} is the $(n+1)$th term so substituting $n+1$ for n into $a_n = \frac{x^n}{n}$ we have

$$a_{n+1} = \frac{x^{n+1}}{n+1}$$

Placing these, $a_n = \frac{x^n}{n}$ and $a_{n+1} = \frac{x^{n+1}}{n+1}$, into $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n+1}\right) \div \left(\frac{x^n}{n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n+1}\right) \times \left(\frac{n}{x^n}\right)\right| \qquad \left[\begin{array}{l}\text{Inverting the Second} \\ \text{Fraction and Multiplying}\end{array}\right]$$

$$= \lim_{n\to\infty}\left|x\left(\frac{n}{n+1}\right)\right| \qquad \left[\text{Simplifying } \frac{x^{n+1}}{x^n} = x\right]$$

$$= |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) \qquad \left[\begin{array}{l}\text{Taking Out } |x| \text{ because} \\ \lim_{n\to\infty}(c\,f(n)) = c\lim_{n\to\infty}(f(n))\end{array}\right]$$

$$= |x|\underbrace{\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)}_{=1} = |x|(1) = |x|$$

So far we have found $L = |x|$ but we need to answer the following questions.

(a) *For what values of x does the series diverge?*

That is when $L > 1$ and from above we have $L = |x|$ so

$$|x| > 1 \quad \text{[Series Diverges]}$$

(b) *For what values of x does the series converge absolutely?*

When $L < 1$ and we know $L = |x|$ and so the given series converges absolutely for

$$|x| < 1 \quad \text{which means } -1 < x < 1$$

The test fails when $L = 1$, that is the values of x which satisfy $|x| = 1$ which means $x = 1$ or $x = -1$.

What happens at $x = 1$?

Substituting $x = 1$ into the given series, $\sum\left(\frac{x^n}{n}\right)$, we have $\sum\left(\frac{1}{n}\right)$ which is the well established harmonic series and this diverges.

What series do we have when $x=-1$?

Substituting $x=-1$ into the given series, $\sum\left(\frac{x^n}{n}\right)$, we have the alternating harmonic series, $\sum\left(\frac{(-1)^n}{n}\right)$, which converges conditionally as shown in the last section.

(c) The series converges conditionally for $x=-1$.

In summary the given series $\sum\left(\frac{x^n}{n}\right)$ converges absolutely for $-1<x<1$ and converges conditionally for $x=-1$ but diverges elsewhere, that is $x\ge 1$ and $x<-1$.

I2 Power Series

Power series is a series of the form

$$c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\ldots=\sum_{n=0}^{\infty}\left(c_nx^n\right)$$

where x is a real variable and c_0, c_1, c_2, c_3,... are constants. These constants are called the coefficients of the series. Notice that the powers of x increase hence the name **power series**. *Can you think of an example of a power series?*

Example 45 above is a power series given by

$$\sum_{n=1}^{\infty}\left(\frac{x^n}{n}\right)=\underbrace{x}_{n=1}+\underbrace{\frac{x^2}{2}}_{n=2}+\underbrace{\frac{x^3}{3}}_{n=3}+\underbrace{\frac{x^4}{4}}_{n=4}+\ldots$$

What are the values of the constants c_0, c_1, c_2, c_3,...*?*

The coefficient c_0 is the constant coefficient, c_1 is the x coefficient, c_2 is the x^2 coefficient, c_3 is the x^3 coefficient etc. In the above case

$$c_0=0 \text{ [Because there is No constant term]},\ c_1=1,\ c_2=\frac{1}{2},\ c_3=\frac{1}{3},\ldots$$

The following are also examples of power series:

(a) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+....=\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$

(b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-....=\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$

(c) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-....=\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$

What are the values of the coefficients c_0, c_1, c_2, c_3, c_4, c_5,... *in each of the above cases?* We have

(a) $c_0=1,\ c_1=1,\ c_2=\frac{1}{2!},\ c_3=\frac{1}{3!},\ c_4=\frac{1}{4!},\ c_5=\frac{1}{5!},$

(b) $c_0=1,\ c_1=0,\ c_2=-\frac{1}{2!},\ c_3=0,\ c_4=\frac{1}{4!},\ c_5=0,$

(c) $c_0 = 0,\ \ c_1 = 1,\ \ c_2 = 0,\ \ c_3 = -\frac{1}{3!},\ \ c_4 = 0,\ \ c_5 = \frac{1}{5!},$

Note that the coefficients, c_n, can be positive, negative or zero.

The above are examples of power series in x. Next we examine convergence of power series. Consider the general power series

$$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots = \sum_{n=0}^{\infty}\left(c_nx^n\right)$$

Although $x \in \mathbb{R}$ but the series may not converge for all x in $\mathbb{R}$. In general it converges for a certain values of x. In Example 45 we found that the given series

$$\sum\left(\frac{x^n}{n}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

converges absolutely for $-1 < x < 1$ and conditionally for $x = -1$ but diverges everywhere else in $\mathbb{R}$ .The interval $-1 \le x < 1$ is called the **interval of convergence**.

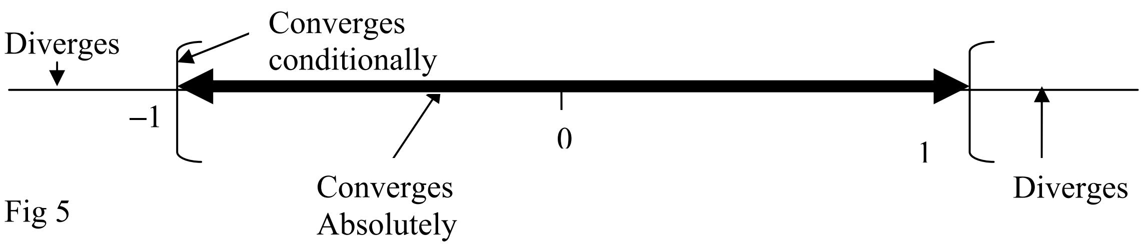


Fig 5

I3 **Radius of Convergence**

If a power series converges in the interval $-R < x < R$ then R is called the **radius of convergence**. The series may converge or diverge at $x = R$ or $x = -R$.

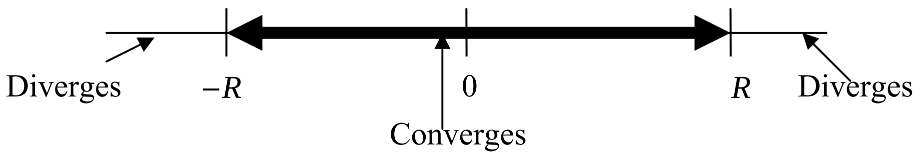


Fig 6

The radius of convergence for Example 45 is $R = 1$ because the interval of convergence is $-1 \le x < 1$.

For example the power series $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$ has radius of convergence $R = +\infty$ and the interval of convergence is $-\infty < x < +\infty$. That is the series converges for all x in $\mathbb{R}$.

The power series $\sum_{n=0}^{\infty}\left(n!x^n\right)$ has radius of convergence $R = 0$ and so it **only** converges at the point $x = 0$.

In this subsection we give a formula definition of radius of convergence.

Proposition (2.23). Let

$$\alpha = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

where the limit exists or $\alpha = +\infty$. Let

$$R = \begin{cases} 1/\alpha & \text{if } \alpha \neq 0 \\ +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \end{cases}$$

then the power series

$$\sum_{n=0}^{\infty}\left(c_n x^n\right)$$

1) Converges absolutely for $|x| < R$

2) Diverges for $|x| > R$

Proof.

We only prove the case for $R = \frac{1}{\alpha}$ ($\alpha \neq 0$). The cases $R = 0$ and $R = +\infty$ is given in Exercise 2i.

Let the nth term be a_n and $(n+1)$th term be a_{n+1} of the power series. Then $a_n = c_n x^n$ and $a_{n+1} = c_{n+1}x^{n+1}$. By applying the above ratio test (2.22) we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right|$$

$$= \lim_{n\to\infty}\left(|x|\left|\frac{c_{n+1}}{c_n}\right|\right) = |x|\left(\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right) = |x|\alpha \quad \left[\text{Because } \alpha = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right]$$

Note that $L = |x|\alpha$. The power series converges absolutely for $L < 1$ which means

$$|x|\alpha < 1 \text{ gives } |x| < \frac{1}{\alpha} = R \qquad [\text{Because } \alpha \neq 0]$$

We have proved part 1).

Where does the series diverge?

When $L > 1$ that is

$$|x|\alpha > 1 \text{ which gives } |x| > \frac{1}{\alpha} = R$$

This proves part 2).

■

I4 **Testing Convergence of Power Series**

We use the ratio test for absolute convergence (2.22) to find the values of x for which the series converges.

Example 46

Determine the values of x for which the following power series converges:

$$\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

Solution

How do we find the values of x for which the given power series converges?

Use the ratio test (2.22) where we need to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

What is a_n equal to in this example?

It is the nth term of the given power series:

$$a_n = \left(\frac{x}{2}\right)^n$$

What is a_{n+1} *equal to in this example?*

Replacing n by $n+1$ into $a_n = \left(\frac{x}{2}\right)^n$ gives

$$a_{n+1} = \left(\frac{x}{2}\right)^{n+1} \quad [(n+1)\text{ th Term}]$$

Substituting $a_n = \left(\frac{x}{2}\right)^n$ and $a_{n+1} = \left(\frac{x}{2}\right)^{n+1}$ into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L = \lim_{n\to\infty}\left|\left(\frac{x}{2}\right)^{n+1} \div \left(\frac{x}{2}\right)^n\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{2^{n+1}}\right) \div \left(\frac{x^n}{2^n}\right)\right| \qquad \left[\text{Because } \left(\frac{x}{2}\right)^{n+1} = \frac{x^{n+1}}{2^{n+1}} \text{ and } \left(\frac{x}{2}\right)^n = \frac{x^n}{2^n}\right]$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}} \times \frac{2^n}{x^n}\right| \qquad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\right| \qquad [\text{Cancelling Common Factors}]$$

$$L = \left|\frac{x}{2}\right| = \frac{|x|}{|2|} = \frac{|x|}{2} \qquad \left[\begin{array}{l}\text{Because } \lim_{n\to\infty}(c) = c\\ \text{and } |2| = 2\end{array}\right]$$

For what values of x *does the series converge absolutely?*

When $L = \frac{|x|}{2} < 1$ that is those x values which satisfy $\frac{|x|}{2} < 1$ gives $|x| < 2$. Hence the given power series converges absolutely for $|x| < 2$ and diverges for $|x| > 2$. The values of x for absolute convergence is $-2 < x < 2$. Remember the ratio test fails when $L = 1$ that is when $|x| = 2$ which means $x = 2$ or $x = -2$. *Does the series converge at* $x = 2$ *and* $x = -2$*?*

First substituting $x = 2$ into the given power series $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$ we have

$$\sum_{n=0}^{\infty}\left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty}(1)^n$$

The series $\sum_{n=0}^{\infty}(1)^n$ diverges because $\lim_{n\to\infty}(1)^n \neq 0$ [Not Zero] that is the nth term does not converge to zero. Hence by (2.6) the series $\sum_{n=0}^{\infty}(1)^n$ diverges. The given power series diverges for $x = 2$.

Similarly substituting $x = -2$ into the given power series $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$ we have

$$\sum_{n=0}^{\infty}\left(\frac{(-2)}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n$$

(2.6) If $\lim_{n\to\infty}(a_n) \neq 0$ then $\sum(a_n)$ diverges

Again the series diverges because $\lim_{n\to\infty}(-1)^n \neq 0$ [Not Zero]. The given power series diverges for $x = -2$.

Summarizing we have the interval of (absolute) convergence is $-2 < x < 2$. Outside this interval the series diverges, that is $x > 2$ and $x < -2$ the given power series diverges.

What is the radius of convergence in this case?

Since $L = \frac{|x|}{2}$ therefore $\alpha = \frac{1}{2}$ because $L = |x|\alpha$ so $R = \frac{1}{1/2} = 2$.

The radius of convergence, R, can be determined directly by $-R < x < R$. Since we have $-2 < x < 2$ therefore $R = 2$.

The next example involves a lot of algebraic simplification and application of limits of sequences. You need to follow this example very carefully.

Example 47

Determine the interval and radius of convergence of the power series

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

Solution.

Again we apply the ratio test (2.22) with $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

What is nth term, a_n, equal to in this case?

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

What is (n+1)th term, a_{n+1}, equal to in this case?

Substituting $n+1$ for n into $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ gives

$$a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$$

Putting these, $a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$ and $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L = \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right) \div \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right) \times \left(\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right)\right| \quad \left[\begin{array}{l}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{array}\right]$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{(-1)^n x^{2n+1}} \times \left(\frac{(2n+1)!}{(2n+3)!}\right)\right| \quad [\text{Collecting Like Terms}]$$

$$= \lim_{n\to\infty}\left|(-1)x^2 \times \left(\frac{(2n+1)!}{(2n+3)!}\right)\right| \quad \left[\text{Cancelling gives } \frac{(-1)^{n+1} x^{2n+3}}{(-1)^n x^{2n+1}} = (-1)x^2\right]$$

But how do we simplify from the last line?

Using $|-1| = 1$ and $|x^2| = x^2$ and

$$\lim_{n\to\infty}\left(c f(n)\right) = c \lim_{n\to\infty}\left(f(n)\right)$$

on the last line we have

$$L = \lim_{n\to\infty}\left|(-1)x^2 \times \left(\frac{(2n+1)!}{(2n+3)!}\right)\right|$$

$$= x^2\left(\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}\right) \qquad (\dagger)$$

How do we simplify the bracket term in (†)?

Well

$$(2n+1)! = 1\times 2\times 3\times \ldots \times 2n \times (2n+1)$$

$$(2n+3)! = 1\times 2\times 3\times \ldots \times 2n \times (2n+1)\times(2n+2)\times(2n+3)$$

$$\frac{(2n+1)!}{(2n+3)!} = \frac{1\times 2\times 3\times \ldots \times 2n \times (2n+1)}{1\times 2\times 3\times \ldots \times 2n \times (2n+1)\times(2n+2)\times(2n+3)}$$

$$= \frac{1}{(2n+2)\times(2n+3)} \quad \left[\begin{array}{l}\text{Cancelling ALL}\\ \text{Common Factors}\end{array}\right]$$

Evaluating this limit gives

$$\lim_{n\to\infty}\left(\frac{1}{(2n+3)(2n+1)}\right) = \lim_{n\to\infty}\left(\frac{1}{4n^2+8n+3}\right) = 0$$

Substituting this, $\lim_{n\to\infty}\left(\frac{(2n+1)!}{(2n+3)!}\right) = 0$, into (†) gives

$$L = x^2(0) = 0$$

For what values of x does the series converge?

For all $x \in \mathbb{R}$ because $L = 0 < 1$ for all $x \in \mathbb{R}$. The interval of (absolute) convergence is $-\infty < x < +\infty$. The radius of convergence is $R = +\infty$.

Until now we have considered a power series about the point $x = 0$. A more general power series is of the form

$$\sum_{n=0}^{\infty} c_n (x-b)^n$$

where b is a real fixed number. We say the power series is expanded about the point b. When $b=0$ then the power series is expanded about 0 as in previous examples.

$$\sum_{n=0}^{\infty} \left(c_n x^n\right) \quad \text{[Power Series about the point } b=0\text{]}$$

How do you write a power series expanded about the point $b=1$*?*

$$\sum_{n=0}^{\infty} c_n (x-1)^n$$

How do you write a power series expanded about the point $b=-2$*?*

$$\sum_{n=0}^{\infty} c_n \left(x-(-2)\right)^n = \sum_{n=0}^{\infty} c_n (x+2)^n$$

Example 48

Determine the interval and radius of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n (x+1)^n}{3^n n^2} \right)$$

Solution.

Note that the given power series is expanded about the point $b=-1$.

We apply the ratio test (2.22) with $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ to find the interval of convergence.

What is a_n *equal to in this case?*

$$a_n = \frac{(-1)^n (x+1)^n}{3^n n^2}$$

What is a_{n+1} *equal to?*

Substituting $n+1$ for n into $a_n = \frac{(-1)^n (x+1)^n}{3^n n^2}$ gives

$$a_{n+1} = \frac{(-1)^{n+1} (x+1)^{n+1}}{3^{n+1} (n+1)^2}$$

Putting $a_n = \frac{(-1)^n (x+1)^n}{3^n n^2}$ and $a_{n+1} = \frac{(-1)^{n+1} (x+1)^{n+1}}{3^{n+1} (n+1)^2}$ into $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ gives

$$L = \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}(x+1)^{n+1}}{3^{n+1}(n+1)^2}\right)\div\left(\frac{(-1)^n(x+1)^n}{3^n n^2}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}(x+1)^{n+1}}{3^{n+1}(n+1)^2}\right)\times\left(\frac{3^n n^2}{(-1)^n(x+1)^n}\right)\right| \quad \begin{bmatrix}\text{Inverting the Second}\\ \text{Fraction and Multiplying}\end{bmatrix}$$

$$= \lim_{n\to\infty}\left|\frac{3^n(-1)^{n+1}(x+1)^{n+1}}{3^{n+1}(-1)^n(x+1)^n}\times\left(\frac{n^2}{(n+1)^2}\right)\right| \quad [\text{Collecting Like Terms}]$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(x+1)}{3}\times\left(\frac{n}{n+1}\right)^2\right| \quad \left[\text{Cancelling gives } \frac{3^n(-1)^{n+1}(x+1)^{n+1}}{3^{n+1}(-1)^n(x+1)^n} = \frac{(-1)(x+1)}{3}\right]$$

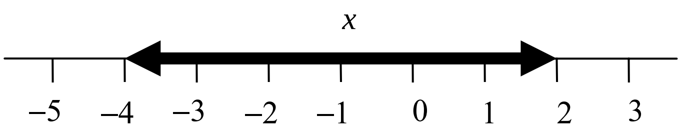
$$= \frac{|x+1|}{3}\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\right) \quad \begin{bmatrix}\text{Because } \lim_{n\to\infty}(c\,f(n)) = c\lim_{n\to\infty}(f(n)),\\ |-1| = 1 \text{ and } |3| = 3\end{bmatrix}$$

$$= \frac{|x+1|}{3}\underbrace{\lim_{n\to\infty}\left(\frac{1}{(1+1/n)}\right)^2}_{=1} \quad \begin{bmatrix}\text{Dividing Numerator}\\ \text{and Denomintor by } n\end{bmatrix}$$

$$= \frac{|x+1|}{3}(1) = \frac{|x+1|}{3}$$

The series converges for $L = \frac{|x+1|}{3} < 1$ which gives $|x+1| < 3$. *What values of* x *satisfy* $|x+1| < 3$*?*

This means that x lies at the centre -1 with distance 3 away from -1.



Fig

Hence the given power series converges absolutely for $-4 < x < 2$. Outside this range the series diverges that is if $x > 2$ and $x < -4$ the given power series diverges.

The test fails for $L = 1$ that is $|x+1| = 3$ [Equal to 3] which means $x = 2$ or $x = -4$.

Does the series converge when $x = 2$ *and* $x = -4$*?*

Substituting the value $x = -4$ into the given power series $\sum_{n=1}^{\infty}\left(\frac{(-1)^n(x+1)^n}{3^n n^2}\right)$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^n(-4+1)^n}{3^n n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n(-3)^n}{3^n n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$\left[\text{Cancelling } \frac{(-1)^n(-3)^n}{3^n} = (-1)^n(-1)^n = (-1)^{2n} = 1\right]$$

This series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ converges absolutely because it is the p-series with $p = 2 > 1$.

Substituting $x = 2$ into the given power series $\sum_{n=1}^{\infty}\left(\frac{(-1)^n (x+1)^n}{3^n n^2}\right)$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^n (2+1)^n}{3^n n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n 3^n}{3^n n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right) \quad \text{[Cancelling } 3^n \text{'s]}$$

Similarly the series $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)$ converges absolutely.

Hence the interval of (absolute) convergence is $-4 \le x \le 2$ and it diverges outside this range, that is for $x > 2$ and $x < -4$ the given power series diverges.

What is the radius of convergence, R, equal to in this case?

Since $\alpha = \frac{1}{3}$ because $L = \frac{|x+1|}{3}$ therefore $R = \frac{1}{1/3} = 3$. Radius of convergence $R = 3$.

Determining values of x for which the power series converges is not a difficult task but it does involve a number of different topics. You need to thoroughly understand evaluation of limits of sequences, algebraic simplification, inequalities and properties of the modulus function.

SUMMARY

A general power series is of the form

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots = \sum_{n=0}^{\infty}\left(c_n x^n\right)$$

We can find the interval of convergence for which the power series converges by using the following ratio test:

Ratio Test for Absolute Convergence (2.22).

Let $\sum(a_n)$ be a series in $\mathbb{R}$ where $\forall n \in \mathbb{N},\quad a_n \neq 0$ $\left[\text{Not Zero}\right]$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

(I) If $L < 1$ then the series $\sum(a_n)$ converges absolutely

(II) If $L > 1$ then the series $\sum(a_n)$ diverges

(III) If $L = 1$ the test fails and we **cannot** conclude whether the series converges absolutely, converges conditionally or diverges.

The radius of convergence, R, is given by

$$R = \begin{cases} 1/\alpha & \text{if } \alpha \neq 0 \\ +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \end{cases}$$

where $\alpha = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ where c_n are the coefficients of the power series.

A power series can be expanded about a point b and it is defined as

$$\sum_{n=0}^{\infty} c_n\left(x-b\right)^n$$