SECTION E **Applications of Determinant**

By the end of this section you will be able to

- apply Cramer's rule to solve linear equations
- determine the number of solutions of a given linear system

In this section we examine an alternative method for solving linear systems of equations. Cramer's rule allows us to find the solution to a system of equations without having to find the inverse, as it is based entirely on determinants.

E1 Cramer's Rule

Gabriel Cramer was born in 1704 in Geneva, Switzerland and by the age of 18 he had received a doctorate on the theory of sound.
In 1724 he attained the chair in mathematics at Academie de Clavin in Geneva and taught geometry and mechanics. Cramer is popularly known in mathematics circles for his contribution to linear algebra, appropriately called 'Cramer's Rule' which was described in his famous book; Introduction to the Analysis of Algebraic Curves. This rule was known to other mathematicians of that era but his superior notation is the reason it is credited to him.

Figure 1 Gabriel Cramer 1704 to 1752

Cramer worked very hard throughout his life, writing books in his spare time and carrying out other editorial work. However, a fall from carriage combined with many years of relentless study contributed to his death at the age of 47.

In this section we first state Cramer's rule and then show how this rule can be used to solve linear systems of equations. Remember a linear system of equations can be written in matrix form as $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is an m by n matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors, that is; an n by 1 matrix. For example we can write the following linear system of equations:

$$\begin{aligned} 2x + 3y - 7z &= 4 \\ 3x - 5y + 11z &= 6 \\ 7x + 6y - z &= 9 \end{aligned}$$

in matrix form as $\mathbf{Ax}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix} 2 & 3 & -7 \\ 3 & -5 & 11 \\ 7 & 6 & -1 \end{pmatrix},\ \mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

How can we solve this linear system of equations?
In previous chapters we discussed various techniques to solve linear systems such as; Gaussian elimination, reduced row echelon form (rref), the 'inverse matrix' method, **LU** factorization etc.
Cramer's rule gives a formula for solving small n by n linear systems by calculating a series of determinants. This often leads to some much simpler arithmetic. On the downside, it becomes inefficient for large systems because it involves evaluating numerous determinants, and there is no easy way to work out the determinant of a large matrix.

Additionally, we can only apply Cramer's rule to an *n* by *n* linear system because we cannot evaluate the determinant of a non-square matrix.
Before we state Cramer's rule we need to introduce some new notation. Let **A** be an *n* by *n* matrix and **b** be a *n* by 1 column vector. The notation $\mathbf{A}_k(\mathbf{b})$ replaces the *k*th column of matrix **A** by the column vector **b**. This means that we have:

$$\mathbf{A}_k(\mathbf{b}) = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}$$

*k*th Column

Therefore $\mathbf{A}_1(\mathbf{b})$ means the first column in matrix **A** is replaced by the *b*'s.

What does $\mathbf{A}_2(\mathbf{b})$ *mean?*

The second column in matrix **A** is replaced by the *b*'s. *What does* $\mathbf{A}_5(\mathbf{b})$ *mean?*

The fifth column in matrix **A** is replaced by the *b*'s.

Cramer's Rule (6.31).
Let **A** be an *n* by *n* matrix with the entries nominated by a_{ij} and **b** be a *n* by 1 column vector. The system of linear equations $\mathbf{Ax} = \mathbf{b}$ has the unique solution:

$$x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\quad x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})},\quad x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})},\ \ldots \text{ and } x_n = \frac{\det(\mathbf{A}_n(\mathbf{b}))}{\det(\mathbf{A})}$$

$\Leftrightarrow$ $\det(\mathbf{A}) \neq 0$ [*not* equal to zero].

How do we prove this result?
We have $\Leftrightarrow$ symbol in the statement which means we need to prove it both ways, $\Rightarrow$ and $\Leftarrow$. First we prove from right to left $\Leftarrow$. *How do we prove this part?*
We assume $\det(\mathbf{A}) \neq 0$ and from this we deduce the above equations for the unknowns x_1, x_2, … , x_n. We are given the linear system $\mathbf{Ax} = \mathbf{b}$ and $\det(\mathbf{A}) \neq 0$ which means that **A** is invertible and by the following result of chapter 1 we have the solution **x** is given by:

(1.36) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

What is inverse matrix $\mathbf{A}^{-1}$ *equal to?*
This was defined in Proposition (6.13):

Proposition (6.13). If $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})} adj(\mathbf{A})$.

We use these Propositions (1.36) and (6.13) to prove Cramer's rule. In order to use Proposition (6.13) we need to know what is meant by *cofactor* and *adjoint* which were defined earlier:

Definition (6.4). The *cofactor* C_{ij} of the entry a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of entry a_{ij}.

Definition (6.9). The adjoint is the cofactor matrix transposed; $adj(\mathbf{A}) = \mathbf{C}^T$.

We are going to use these definitions and propositions to prove Cramer's rule. If you are not confident in applying these then you will need to go back and see exactly what they mean before embarking on the proof of this result.

Proof.

$(\Leftarrow)$. We assume that $\det(\mathbf{A}) \neq 0$ and prove the result for x_1 (x one) and then generalize to the remaining unknowns. From the given linear system $\mathbf{Ax} = \mathbf{b}$, we have by the inverse matrix method, Proposition (1.36), $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ where

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A}) \qquad \text{[This was defined in Proposition (6.13)]}$$

What does $adj(\mathbf{A})$ represent?

$adj(\mathbf{A})$ is the adjoint matrix and by (6.9) this is the cofactor matrix transposed, that is

$$adj(\mathbf{A}) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \text{ where } C\text{'s are cofactors}$$

By Definition (6.4) the cofactor of an entry is the place sign times the determinant of the remaining matrix after deleting the row and column containing that entry. The cofactors C's are given by:

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (-1)^{1+1} \det \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (\dagger)$$

$$C_{21} = (-1)^{2+1} \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (-1)^{2+1} \det \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (\dagger\dagger)$$

$$\vdots$$

$$C_{n1} = (-1)^{n+1} \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (-1)^{n+1} \det \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{pmatrix} \qquad (\dagger\dagger\dagger)$$

By applying Proposition (6.13) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det(\mathbf{A})}\left(adj(\mathbf{A})\right)\mathbf{b}$ we have

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det(\mathbf{A})} \begin{pmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{pmatrix} \quad \left[\begin{array}{l}\text{Multiplying out the } \mathbf{C} \text{ matrix} \\ \text{and the } \mathbf{b} \text{ vector with} \\ \text{row by column multiplication}\end{array}\right]$$

We show the result for x_1 and the proofs of x_2, $x_3, \cdots$ and x_n are similar.

Equating x_1 on the Left with the first entry on the Right in the above gives :

$$x_1 = \frac{1}{\det(\mathbf{A})}\left(b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1}\right)$$

$$\underbrace{=}_{\substack{\text{Substituting the}\\\text{determinant}\\\text{given above of}\\\text{the cofactor}\\\text{matrices}}} \frac{1}{\det(\mathbf{A})}\left[\underbrace{b_1(-1)^{1+1}\det\begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn}\end{pmatrix}}_{\text{By } (\dagger)} + \underbrace{b_2(-1)^{2+1}\det\begin{pmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn}\end{pmatrix}}_{\text{By } (\dagger\dagger)} + \cdots \\ \cdots + \underbrace{b_n(-1)^{n+1}\det\begin{pmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)}\end{pmatrix}}_{\text{By } (\dagger\dagger\dagger)}\right] \quad (*)$$

What do we need to prove?

$x_1 = \dfrac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \dfrac{1}{\det(\mathbf{A})}\det(\mathbf{A}_1(\mathbf{b}))$. We have $\dfrac{1}{\det(\mathbf{A})}$ in (*) but we need to show the expression in the large square brackets on the Right is equal to $\det(\mathbf{A}_1(\mathbf{b}))$. *How?*

Need to find $\det(\mathbf{A}_1(\mathbf{b}))$. *What is* $\det(\mathbf{A}_1(\mathbf{b}))$ *equal to?*

Well $\mathbf{A}_1(\mathbf{b})$ is the matrix **A** but the first column is replaced by the column vector **b**, that is:

$$\mathbf{A}_1(\mathbf{b}) = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

We can evaluate the determinant of this matrix by expanding along the first column,

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} \boxed{b_1} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

Expanding along this column.

$$= (-1)^{1+1}b_1\det\begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn}\end{pmatrix} + (-1)^{2+1}b_2\det\begin{pmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn}\end{pmatrix} + \cdots$$

$$\cdots + (-1)^{n+1}b_n\det\begin{pmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)}\end{pmatrix}$$

What do you notice about this last expression?

This is identical to the above expression in the square brackets in (*). By (*) we have

$$\det(\mathbf{A}_1(\mathbf{b})) = b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1}$$

We have

$$x_1 = \frac{1}{\det(\mathbf{A})}(b_1C_{11} + \cdots + b_nC_{n1}) = \frac{1}{\det(\mathbf{A})}\det(\mathbf{A}_1(\mathbf{b}))$$

Similarly we can show that for $j = 2, 3, \cdots, n$

$$x_j = \frac{1}{\det(\mathbf{A})}\det(\mathbf{A}_j(\mathbf{b}))$$

Hence by substituting each j value we have

$$x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})},\ x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}, \ldots \text{ and } x_n = \frac{\det(\mathbf{A}_n(\mathbf{b}))}{\det(\mathbf{A})}$$

$(\Rightarrow)$. If we have

$$x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\ x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})},\ x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}, \ldots \text{ and } x_n = \frac{\det(\mathbf{A}_n(\mathbf{b}))}{\det(\mathbf{A})}$$

Then clearly $\det(\mathbf{A}) \neq 0$. This completes our proof.

■

We can apply Cramer's rule to a particular linear system as the next example shows.

Example 21

Solve the following linear system by using Cramer's rule:

$$\begin{aligned} 2x - y - 5z &= 16 \\ -7x + 2y + z &= 7 \\ 5x - 3y - 3z &= 1 \end{aligned}$$

<u>Solution</u>

We can write this in matrix form $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -5 \\ -7 & 2 & 1 \\ 5 & -3 & -3 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 16 \\ 7 \\ 1 \end{pmatrix}$$

Applying Cramer's rule (provided $\det(\mathbf{A}) \neq 0$) we have

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\ y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} \text{ and } z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} \qquad (*)$$

What is $\det(\mathbf{A})$ *equal to?*

$$\det(\mathbf{A}) = \det\begin{pmatrix} 2 & -1 & -5 \\ -7 & 2 & 1 \\ 5 & -3 & -3 \end{pmatrix} = 2\det\begin{pmatrix} 2 & 1 \\ -3 & -3 \end{pmatrix} - (-1)\det\begin{pmatrix} -7 & 1 \\ 5 & -3 \end{pmatrix} - 5\det\begin{pmatrix} -7 & 2 \\ 5 & -3 \end{pmatrix}$$

$$= 2(-6+3) + (21-5) - 5(21-10)$$

$$= -45$$

Hence $\det(\mathbf{A}) = -45$. *What else do we need to find?*

By equations in (*) we need to find $\det(\mathbf{A}_1(\mathbf{b}))$, $\det(\mathbf{A}_2(\mathbf{b}))$ and $\det(\mathbf{A}_3(\mathbf{b}))$ in order to determine x, y and z respectively. *What is* $\det(\mathbf{A}_1(\mathbf{b}))$ *equal to?*

$\mathbf{A}_1(\mathbf{b})$ is the matrix $\mathbf{A}$ but with the first column replaced by $\mathbf{b}=\begin{pmatrix}16\\7\\1\end{pmatrix}$. Hence $\det(\mathbf{A}_1(\mathbf{b}))$ is the determinant of this matrix:

$$\det(\mathbf{A}_1(\mathbf{b}))=\det\begin{pmatrix}16 & -1 & -5\\ 7 & 2 & 1\\ 1 & -3 & -3\end{pmatrix}$$

Replacing the first column by **b**.

$$=16\det\begin{pmatrix}2 & 1\\ -3 & -3\end{pmatrix}+1\det\begin{pmatrix}7 & 1\\ 1 & -3\end{pmatrix}-5\det\begin{pmatrix}7 & 2\\ 1 & -3\end{pmatrix}$$
$$=16(-6+3)+(-21-1)-5(-21-2)=45$$

What is $\det(\mathbf{A}_2(\mathbf{b}))$ *equal to?*

$\mathbf{A}_2(\mathbf{b})$ is the matrix $\mathbf{A}$ but with the second column replaced by $\mathbf{b}$ and $\det(\mathbf{A}_2(\mathbf{b}))$ is the determinant of this matrix:

$$\det(\mathbf{A}_2(\mathbf{b}))=\det\begin{pmatrix}2 & 16 & -5\\ -7 & 7 & 1\\ 5 & 1 & -3\end{pmatrix}$$

Replacing the second column by **b**.

$$=2\det\begin{pmatrix}7 & 1\\ 1 & -3\end{pmatrix}-16\det\begin{pmatrix}-7 & 1\\ 5 & -3\end{pmatrix}-5\det\begin{pmatrix}-7 & 7\\ 5 & 1\end{pmatrix}$$
$$=2(-21-1)-16(21-5)-5(-7-35)=-90$$

What is $\det(\mathbf{A}_3(\mathbf{b}))$ *equal to?*

Similarly $\mathbf{A}_3(\mathbf{b})$ is the matrix $\mathbf{A}$ but with the third column replaced by $\mathbf{b}$ and $\det(\mathbf{A}_3(\mathbf{b}))$ is

$$\det(\mathbf{A}_3(\mathbf{b}))=\det\begin{pmatrix}2 & -1 & 16\\ -7 & 2 & 7\\ 5 & -3 & 1\end{pmatrix}$$

Replacing the third column by **b**.

$$=2\det\begin{pmatrix}2 & 7\\ -3 & 1\end{pmatrix}+1\det\begin{pmatrix}-7 & 7\\ 5 & 1\end{pmatrix}+16\det\begin{pmatrix}-7 & 2\\ 5 & -3\end{pmatrix}$$
$$=2(2+21)+(-7-35)+16(21-10)=180$$

Substituting $\det(\mathbf{A})=-45$, $\det(\mathbf{A}_1(\mathbf{b}))=45$, $\det(\mathbf{A}_2(\mathbf{b}))=-90$ and $\det(\mathbf{A}_3(\mathbf{b}))=180$:

$$x=\frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\ y=\frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}\text{ and } z=\frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} \qquad (*)$$

gives

$$x=\frac{45}{-45}=-1,\ y=\frac{-90}{-45}=2\text{ and } z=\frac{180}{-45}=-4$$

Hence the unique solution of the linear system is $x=-1$, $y=2$ and $z=-4$.

E2 Linear Systems of Equations

Remember linear algebra is the study of linear equations and in this subsection we examine the relationship between the solutions of a system and the determinant of its' matrix.

From section C we know that the matrix $\mathbf{A}$ is invertible $\Leftrightarrow$ $\det(\mathbf{A}) \neq 0$. (Theorem (6.26)).

This means that matrix $\mathbf{A}$ being invertible is equivalent to $\det(\mathbf{A}) \neq 0$. Hence we can add this $\det(\mathbf{A}) \neq 0$ to Theorem (1.38) of chapter 1:

Theorem (6.32). Let $\mathbf{A}$ be an n by n matrix, then the following 6 statements are equivalent:
(a) The matrix $\mathbf{A}$ is invertible (non-singular).
(b) The linear system $\mathbf{Ax} = \mathbf{O}$ only has the trivial solution $\mathbf{x} = \mathbf{O}$.
(c) The reduced row echelon form of the matrix $\mathbf{A}$ is the identity matrix $\mathbf{I}$.
(d) $\mathbf{A}$ is a product of elementary matrices.
(e) $\mathbf{Ax} = \mathbf{b}$ has a unique solution.
(f) $\det(\mathbf{A}) \neq 0$.

We have added statement (f) to Theorem (1.38) of chapter 1.

What can we conclude about the linear system $\mathbf{Ax} = \mathbf{b}$ if $\det(\mathbf{A}) = 0$?

Two things

1. If $\mathbf{b} \neq \mathbf{O}$ [Not Zero] then $\mathbf{Ax} = \mathbf{b}$ has an *infinite* number or *no* solutions.
2. If $\mathbf{b} = \mathbf{O}$ then $\mathbf{Ax} = \mathbf{O}$ has an infinite number of solutions. Clearly in this case we have the trivial solution $\mathbf{x} = \mathbf{O}$ which means $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \cdots,\ x_n = 0$ and an infinite number of other solutions.

Example 22

Which of the following linear systems have a non-trivial solution?

(a)
$$\begin{aligned} 2x - y - 5z &= 0 \\ -7x + 2y + z &= 0 \\ 5x - 3y - 3z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 6x + 7y + 8z &= 0 \end{aligned}$$

<u>Solution</u>

(a) Note that in this case the matrix of coefficients is identical to the one in Example 21. Thus by the above Example 21 we have $\mathbf{Ax} = \mathbf{O}$ where

$$\det(\mathbf{A}) = -45$$

Do we have a non-trivial solution to the given linear system?

No because $\det(\mathbf{A}) \neq 0$ [Not Zero] therefore by the above Proposition (6.32) part (b) we *only* have the trivial solution:

$$x = 0,\ y = 0 \text{ and } z = 0$$

(b) We have $\mathbf{Ax} = \mathbf{O}$ where

$$\det(\mathbf{A}) = \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$
$$= (45 - 48) - 2(36 - 42) + 3(32 - 35) = 0$$

What can we conclude about the given linear system?

Since $\det(\mathbf{A}) = 0$ so by statement 2 above we can say that the given linear system has an infinite number of solutions and so has non-trivial solutions.

Note that we do *not* have to find them.

Example 23

For what values of k will the following system have
(i) a unique solution? *(ii) an infinite number or no solution?*

$$\begin{matrix} x & + & y & + & kz & = & 3 \\ x & + & ky & + & z & = & 3 \\ kx & - & y & + & z & = & 1 \end{matrix}$$

Solution

Writing the given linear system in matrix form we have

$$\mathbf{Ax}=\mathbf{b} \text{ where } \mathbf{A}=\begin{pmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & -1 & 1 \end{pmatrix}, \quad \mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

The determinant of the matrix **A** is given by

$$\det(\mathbf{A})=\begin{pmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & -1 & 1 \end{pmatrix}=1(k+1)-1(1-k)+k\left(-1-k^2\right)$$
$$=k+1-1+k-k-k^3$$
$$=k-k^3=k\left(1-k^2\right)$$

(i) *Under what conditions do we have a unique solution?*

It is where $\det(\mathbf{A})\neq 0$. Thus we have a unique solution provided

$$\det(\mathbf{A})=k\left(1-k^2\right)\neq 0 \text{ which occurs when } k\neq 0 \text{ or } k^2-1\neq 0$$

Thus we have unique solution provided $k\neq 0$ or $k\neq \pm 1$.

(ii) *Under what conditions do we have no or an infinite number of solutions?*

$$\det(\mathbf{A})=k\left(1-k^2\right)=0 \text{ which gives } k=0 \text{ or } k=\pm 1$$

Thus we have *no* or an *infinite* number of solutions provided $k=0$ or $k=\pm 1$.

SUMMARY

Cramer's Rule (6.31). The linear system $\mathbf{Ax}=\mathbf{b}$ has the unique solution:

$$x_1=\frac{\det\left(\mathbf{A}_1(\mathbf{b})\right)}{\det(\mathbf{A})},\quad x_2=\frac{\det\left(\mathbf{A}_2(\mathbf{b})\right)}{\det(\mathbf{A})},\quad x_3=\frac{\det\left(\mathbf{A}_3(\mathbf{b})\right)}{\det(\mathbf{A})},\ \ldots \text{ and } x_n=\frac{\det\left(\mathbf{A}_n(\mathbf{b})\right)}{\det(\mathbf{A})}$$

$\Leftrightarrow$ $\det(\mathbf{A})\neq 0$.