SECTION D **Composite Functions**

By the end of this section you will be able to

- understand what is meant by a composite function
- find composition of functions
- combine functions by addition, subtraction, multiplication and division
- understand what is meant by equality of functions

D1 Definition of Composite Function

What does the word 'composite' mean in everyday language?

Something which is made up of separate parts. A composite function is a function which is made up of several functions.

A composite function is also called a '**function of a function'**.

Consider functions $g : A \to B$ and $f : B \to C$ illustrated below:

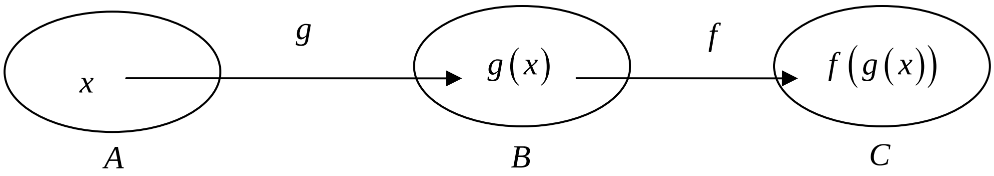

Fig 26

Let x be an element in set A. Then the image of x under the function g given by $g(x)$ is in set B. Note that set B is the domain (start) of the function f. The image of $g(x)$ under the function f is given by $f(g(x))$ and it is in the set C.

The function from set A to set C which assigns each element x in A the element $f(g(x))$ in C is called the composition of the functions f and g. This is normally denoted by $f \circ g$ and verbally stated as 'f composite g'. Hence for all x in the domain A we have

(3.6) $$(f \circ g)(x) = f(g(x))$$

This is the function $f \circ g$. Note that we carry out the operation g **first** and then apply the function f to this result.

Example 23

Let $g : A \to B$ and $f : B \to C$ be defined by the following diagram:

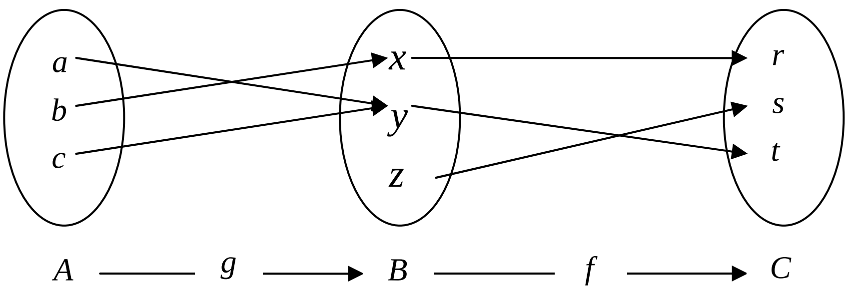

Fig 27

Compute $(f \circ g)(a)$, $(f \circ g)(b)$ and $(f \circ g)(c)$.

Solution
By applying
(3.6) $(f \circ g)(x) = f(g(x))$
and following the arrows in Fig 27 we have

$$(f \circ g)(a) = f(g(a))$$
$$= f(y) = t \qquad [\text{Because } g(a) = y]$$
$$(f \circ g)(b) = f(g(b)) \qquad [\text{By (3.6)}]$$
$$= f(x) = r \qquad [\text{Because } g(b) = x]$$
$$(f \circ g)(c) = f(g(c)) \qquad [\text{By (3.6)}]$$
$$= f(y) = t \qquad [\text{Because } g(c) = y]$$

Example 24
Let $g: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

i. Determine $(f \circ g)(3)$
ii. Determine $(g \circ f)(3)$
iii. Find the function $(f \circ g)(x)$ and $(g \circ f)(x)$ stating the domain and codomain in each case.
What do you notice about your results?

Solution
i. *What does* $(f \circ g)(3)$ *mean?*
Means evaluate $g(3)$ first and then take f of this result. This is add 1 to the argument 3 first and then square your result. By
(3.6) $(f \circ g)(x) = f(g(x))$
we have

$$(f \circ g)(3) = f(g(3))$$
$$= f(3+1) \qquad [\text{Because } g(x) = x+1]$$
$$= f(4) = 4^2 = 16 \qquad [\text{Because } f(x) = x^2]$$

ii. Similarly we have

$$(g \circ f)(3) = g(f(3)) \qquad [\text{Definition of } \circ]$$
$$= g(3^2) \qquad [\text{Because } f(x) = x^2]$$
$$= g(9) = 9 + 1 = 10 \qquad [\text{Because } g(x) = x+1]$$

iii. *What is* $(f \circ g)(x)$ *equal to?*
Again by
(3.6) $(f \circ g)(x) = f(g(x))$
we have

$$(f \circ g)(x) = f(g(x))$$
$$= f(x+1) \qquad [\text{Because } g(x) = x+1]$$
$$= (x+1)^2 = x^2 + 2x + 1 \qquad [\text{Because } f(x) = x^2]$$

Similarly we have

$$(g \circ f)(x) = g(f(x)) \qquad [\text{Definition of } \circ]$$
$$= g(x^2) \qquad [\text{Because } f(x) = x^2]$$
$$= x^2 + 1 \qquad [\text{Because } g(x) = x+1]$$

The domain and codomain for both $(f \circ g)(x)$ and $(g \circ f)(x)$ is $\mathbb{R} \to \mathbb{R}$.

Notice that

$$(f \circ g)(3) = 16 \quad \text{and} \quad (g \circ f)(3) = 10$$

Also

$$(f \circ g)(x) = x^2 + 2x + 1 \quad \text{and} \quad (g \circ f)(x) = x^2 + 1$$

Hence we conclude that

$$f \circ g \neq g \circ f \qquad [\text{Not Equal}]$$

The above result is important so we give it a reference number:

(3.7) $\quad f \circ g \neq g \circ f \qquad$ [Not Equal]

where f and g are functions such that the **range** (arrival) of g is in the domain (start) of f.

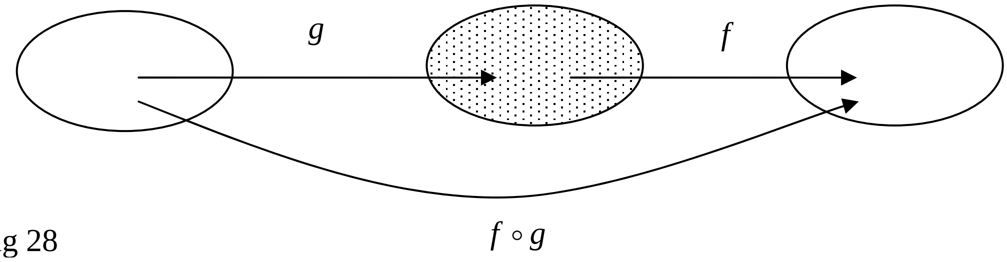

Fig 28

The shaded area in figure 28 is the range of g and the domain of f.

For composite functions such as $f \circ g$ care must be taken that the range of g is contained in the domain of f.

A **real function** is a function whose **domain and codomain are real numbers**. The functions f and g defined in Example 24 are real functions but the functions defined in Example 23 are **not** unless the symbols $a, b, c, x, \ldots$ represent real numbers.

Example 25

Let the functions f and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = 1 - x^2 \quad \text{and} \quad f(x) = \sqrt{x}$$

What is largest domain of f so that $f \circ g$ is a real function?

Solution

Computing

$$(f \circ g)(x) = f(g(x)) \qquad \text{[Definition of } \circ]$$
$$= f(1-x^2) \qquad \left[\text{Because } g(x) = 1-x^2\right]$$
$$= \sqrt{1-x^2} \qquad \left[\text{Because } f(x) = \sqrt{x}\right]$$

For what values of x is $\sqrt{1-x^2}$ real?
Clearly for $-1 \le x \le 1$ and that is all the real numbers between -1 to 1 inclusive. This is the largest domain of f for $f \circ g$ to be a real function. We can write this as the set $A = \{x \mid x \in \mathbb{R} \text{ and } -1 \le x \le 1\}$ or as the interval $[-1, 1]$.

What is a domain and codomain of the function f ?
We could have the domain and codomain of the function f as follows:
$f : A \to \mathbb{R}$ where the set A is the largest domain so that $f \circ g$ is a real function.

Note that the range (arrival) of g **must** be in the domain (start) of f for the composite function $f \circ g$. In the above example the codomain of g is the set of all the real numbers, $\mathbb{R}$, but the range is only the real numbers between -1 to 1 inclusive. This can be illustrated as:

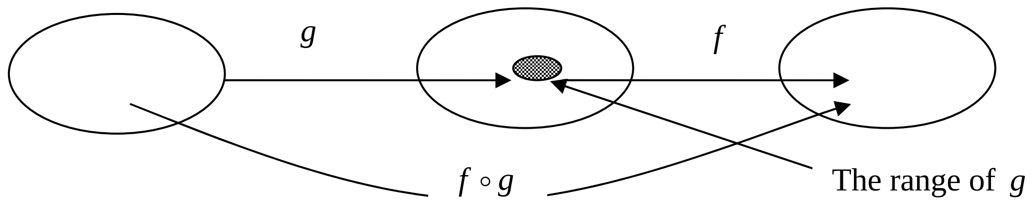

Fig 29

*What happens if the range of g is **not** in the domain of f ?*
Then some elements will **not** be defined by the composite function $f \circ g$. For example, let

$$g : \mathbb{R} \to \mathbb{R} \text{ be defined by } g(x) = x - 5$$

and

$$f : \mathbb{R}^+ \to \mathbb{R} \text{ be defined by } f(x) = x^2$$

What is the value of $(f \circ g)(2)$?

$$(f \circ g)(2) = f(g(2))$$
$$= f(-3) \qquad \left[\text{Because } g(2) = 2 - 5 = -3\right]$$

But -3 is **not** in the domain of f because the domain of f is all the positive real numbers, $\mathbb{R}^+$. Hence $f(-3)$ does **not** exist or $(f \circ g)(2)$ is **not** defined.

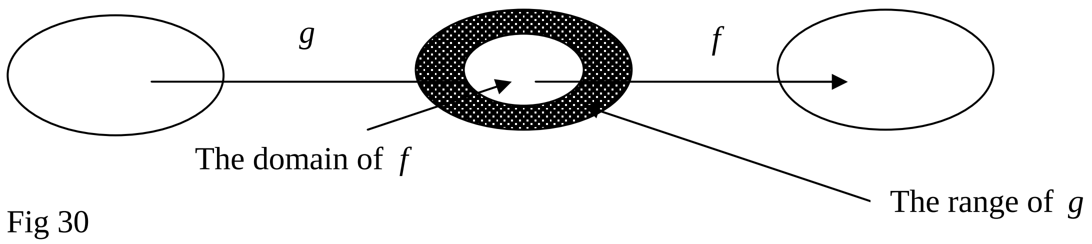

Fig 30

The shaded part in Fig 30 is the range of g which is not in the domain of f. Hence the elements in the shaded part will never be transformed by the function f.

D2 Examples of Composition of Functions

In this subsection we compute the composition of functions.

Example 26

Let the functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be defined by the formulae

$$f(x)=x^2 \text{ and } g(x)=\sin(x)$$

Find a formula for

(a) $f\circ g$ (b) $g\circ f$

Solution (a) Computing $f\circ g$:

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) && [\text{Definition of } \circ]\\ &= f(\sin(x)) && [\text{Because } g(x)=\sin(x)]\\ &= (\sin(x))^2 && [\text{Because } f(x)=x^2]\\ &= \sin^2(x) && \left[\text{Remember } (\sin(x))^2=\sin^2(x)\right]\end{aligned}$$

(b) Computing $g\circ f$:

$$\begin{aligned}(g\circ f)(x) &= g(f(x)) && [\text{Definition of } \circ]\\ &= g(x^2) && [\text{Because } f(x)=x^2]\\ &= \sin(x^2) && [\text{Because } g(x)=\sin(x)]\end{aligned}$$

Of course we can have composition of more than 2 functions as the next example shows.

Example 27

Let the function $f:\mathbb{R}\to\mathbb{R}$ be defined by the formula

$$f(x)=\sin(x)$$

Find a formula for $f\circ f\circ f\circ f$.

Solution

This is not difficult but the notation with all the brackets maybe confusing.

$$\begin{aligned}(f\circ f\circ f\circ f)(x) &= (f\circ f\circ f)[f(x)] && [\text{Definition of } \circ]\\ &= (f\circ f\circ f)[\sin(x)] && [\text{Because } f(x)=\sin(x)]\\ &= (f\circ f)(f[\sin(x)])\\ &= (f\circ f)(\sin[\sin(x)]) && [\text{Because } f=\sin]\\ &= (f)\circ[f(\sin[\sin(x)])]\\ &= (f)[\sin(\sin[\sin(x)])]\\ &= \sin[\sin(\sin[\sin(x)])]\end{aligned}$$

It is very easy to get confused with all the brackets that is why we used a combination of brackets.

The composition of functions is **not** the multiplication of functions but is the function of a function. In next section we describe the multiplication, division etc of functions.

D3 Other Combination of Functions

We can define addition, subtraction, multiplication and division of functions as follows:

(3.8) $$(f \pm g)(x) = f(x) \pm g(x)$$

(3.9) $$f \cdot g(x) = f(x) \times g(x)$$

(3.10) $$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \quad \text{provided } g(x) \neq 0$$

The domain of $f \pm g$ and $f \cdot g$ is the set

$$\{\text{Domain of } f\} \cap \{\text{Domain of } g\}$$

The domain of $\dfrac{f}{g}$ is

$$\{\text{Domain of } f\} \cap \{\text{Domain of } g\} \cap \{x \mid g(x) \neq 0\}$$

Example 27

Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f(x) = x - 1 \text{ and } g(x) = 3x + 5$$

Determine a formula for

(i) $f + g$ (ii) $f - g$ (iii) $g - 5$ (iv) $f \cdot g$ (v) $\dfrac{f}{g}$

Solution

(i) By (3.8) we have

$$\begin{aligned} f + g &= (f + g)(x) \\ &= f(x) + g(x) \\ &= (x - 1) + (3x + 5) \\ &= 4x + 4 \end{aligned}$$

(ii) Applying (3.8) again we have

$$\begin{aligned} f - g &= (f - g)(x) \\ &= f(x) - g(x) \\ &= (x - 1) - (3x + 5) \\ &= -2x - 6 \end{aligned}$$

(iii) Using (3.8) with a constant function, 5, taken away from g:

$$\begin{aligned} g - 5 &= g(x) - 5 \\ &= (3x + 5) - 5 \\ &= 3x \end{aligned}$$

(iv) By (3.9) we have

$$\begin{aligned} f \cdot g &= (f \cdot g)(x) \\ &= f(x) \times g(x) \\ &= (x - 1)(3x + 5) \\ &= 3x^2 + 2x - 5 \qquad [\text{Expanding Brackets}] \end{aligned}$$

(v) By (3.10) we have

$$\frac{f}{g} = \frac{f(x)}{g(x)}$$

$$= \frac{x-1}{3x+5} \qquad \text{provided } 3x+5 \neq 0$$

The domain in each case is all the real numbers, $\mathbb{R}$, apart from (v). *What is the domain in this case?*

Is all the real numbers excluding where $3x+5=0$ that is $x=-\frac{5}{3}$.

D4 Equal Functions

What are equal functions?

Functions f and g are equal if and only if both functions, f and g, have the same domain and also for every x in the domain we have

(3.11) $\qquad f(x) = g(x)$

This is normally written as $f = g$.

Example:

Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x) = x+1 \text{ and } g(y) = y+1$$

Then functions f and g are equal, $f = g$. Remember the symbols x and y are *dummy* variables and just used to define a formula for a function.

Example:

Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{Z}\to\mathbb{Z}$ be given by

$$f(x) = x+1 \text{ and } g(x) = x+1$$

The formulae for the functions f and g are identitical but the domains are different, $\mathbb{R}$ and $\mathbb{Z}$ respectively. These functions are **not** equal, that is $f \neq g$. For functions to be equal the domains need to be the same.

Example 26

Let the functions $f: A\to B$ and $g: A\to B$ where

$$A=\{1,\ 2\} \text{ and } B=\{3,\ 4\}$$

be defined by the formulae

$$f(x) = x+2 \text{ and } g(x) = x^2-2x+4$$

Show that $f = g$.

Solution

What is the domain of the given functions f and g?

Domain is the set $A=\{1,\ 2\}$. That is the domain of f and g are the two integers 1 and 2. Substituting $x=1$ and $x=2$ into the given formulae for functions:

$$f(x) = x+2 \text{ and } g(x) = x^2-2x+4$$

yields

$$f(1) = 1+2 = 3 \quad \text{and} \quad g(1) = 1^2-(2\times 1)+4 = 3$$

$$f(2) = 2+2 = 4 \quad \text{and} \quad g(2) = 2^2-(2\times 2)+4 = 4$$

Since $f(x)=g(x)$ for all x values in the domain we conclude $f=g$.

SUMMARY

The composition of functions is a **function of a function** and is denoted by the symbol $\circ$ and defined by

$$(f\circ g)(x)=f(g(x))$$

Remember the order is important. For $(f\circ g)(x)$ we evaluate $g(x)$ first and then apply the function f to this result.

The range of g must be in the domain of f. Also

$$f\circ g\neq g\circ f \qquad [\text{Not Equal}]$$

Other combinations are

(3.8) $$(f\pm g)(x)=f(x)\pm g(x)$$

(3.9) $$f\cdot g(x)=f(x)\times g(x)$$

(3.10) $$\frac{f}{g}(x)=\frac{f(x)}{g(x)} \quad \text{provided } g(x)\neq 0$$

Functions f and g are equal, $f=g$, if and only if for all x in the domain

$$f(x)=g(x)$$