Chapter 2 : Infinite Series

Section **An Introduction to Infinite Series**

By the end of this section you will be able to
- understand what is meant by convergence and divergence of infinite series
- recognise a geometric series
- determine whether a given geometric series converges and determine its sum

A1 **Introduction**

A lot of students are confused between sequences and series.
A sequence is of the form $a_1,\ a_2,\ a_3,\ a_4,...$
A series is of the form $a_1 + a_2 + a_3 + a_4 + ...$
For example 1, 2, 3, 4,... is an example of a sequence but

$$1+2+3+4+...$$

is an example of a series. We use the Greek letter , $\sum$, pronounced sigma for writing the series in compact form. For example

$$1+2+3+4+...$$

is written as $\sum_{n=1}^{\infty} n$, that is

$$1+2+3+4+...=\sum_{n=1}^{\infty} n$$

How would you write the following in $\sum$ *, sigma, notation?*

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$
2) $1+2+4+8+16+...$
3) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...$

1) can be a written as $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...=\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$
2) can be written as $1+2+4+8+16+...=\sum_{n=0}^{\infty}\left(2^n\right)$ [Start at $n=0$ because $2^0=1$]
3) can be written as $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...=\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)$

These 1), 2) and 3) are all examples of infinite series. *What does infinite series mean?*
A series which has an unlimited number of terms is called an infinite series.
A sequence $S_1,\ S_2,\ S_3,\ ...\ ,\ S_n$ defined by

$S_1 = a_1$ [Sum of the First Term]

$S_2 = a_1 + a_2$ [Sum of the First 2 Terms]

$S_3 = a_1 + a_2 + a_3$ [Sum of the First 3 Terms]

“

“

$S_n = a_1 + a_2 + a_3 + ... + a_n$ [Sum of the First n Term]

are called the nth partial sum of the series. These are examples of a finite series.
What does finite series mean?
A series which has a finite number of terms is called a **finite** series.
In this chapter we consider infinite series and are interested in the convergence of these infinite series. *What does convergent series mean?*
Let S_n be the nth partial sum of an infinite series which means it is the sum of the first n terms:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

If S_n approaches a limit L as the number of terms, n, approaches infinity then we say the infinite series converges. That is an infinite series is convergent if

$$\lim_{n \to \infty} (S_n) = L$$

where L is a real number. If S_n converges to a real number L as n goes to infinity then the series is said to converge to L and L is also called the sum of the series.
For example

$$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \ldots = \frac{1}{4}$$

This means adding infinitely many terms $\frac{1}{5}$, $\frac{1}{5^2}$, $\frac{1}{5^3}$, $\frac{1}{5^4}$, etc gives the sum $\frac{1}{4}$.
In the table below we have evaluated the sum in the right column for corresponding number of terms in the left column.

n	$\sum_{k=1}^{n} \left(\frac{1}{5^k} \right)$
1	0.2000000000
5	0.2499200000
10	0.2499999744
20	0.2500000000

TABLE 1

We can sum a finite number of the terms by using a computer algebra system such as MAPLE.
If $\lim_{n \to \infty} (S_n)$ does not approach a real number as n goes to infinity then the series is said to **diverge**.
For example $\sum_{k=1}^{\infty} \left(\frac{1}{k} \right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges
What does this mean?
If we take enough terms of the series then we can make the sum as large as we like.

n	$\sum_{k=1}^{n} \left(\frac{1}{k} \right)$
1	1
10	2.928968254
100	5.187377518
1 000	7.485470861
10 000	9.787606036
100 000	12.09014612
1 000 000	14.39272672

TABLE 2

In the table above the left hand column is the number of terms (n) and the right column gives the corresponding sum:

That is the sum of the first 100 terms of the series $\sum_{k=1}^{n}\left(\frac{1}{k}\right)$ is just over 5 and the sum of the first million terms (1 000 000) is about 14.4.

Generally for the rest of this chapter we test the convergence of given infinite series. For this test we need to know certain results regarding sequences to show that a series converges. For example we use the following result to test convergence:

(2.1) $$\lim_{n\to\infty}\left(x^n\right)=0 \quad \text{if } |x|<1$$

If $|x|\geq 1$ then $\lim_{n\to\infty}\left(x^n\right)$ does **not** converge.

In testing a given series for convergence we first write nth partial sum, S_n, and then determine $\lim_{n\to\infty}\left(S_n\right)$ by using our results of sequences from the last chapter.

To get a feel for a series it is good practice to write out the first few terms of the series.

Example 1

Show that $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)$ converges and find its sum.

Solution.

We first consider the nth partial sum, S_n, of $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)$. *How can we write* S_n *?*

Well $S_n=\sum_{k=1}^{n}\left(\frac{1}{2^k}\right)$ is the sum of the first n terms and is written as:

$$S_n=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^n} \qquad (*)$$

Multiplying both sides of (*) by 2 gives

$$2S_n=2\left(\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^n}\right)$$

$$=\frac{2}{2}+\frac{2}{2^2}+\frac{2}{2^3}+\frac{2}{2^4}+\ldots+\frac{2}{2^n} \qquad \left[\begin{array}{l}\text{Multiplying each term}\\ \text{in bracket by 2}\end{array}\right]$$

$$2S_n=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}} \qquad \left[\begin{array}{l}\text{Simplifying by}\\ \text{Rules of Indices}\end{array}\right] \qquad (**)$$

Subtracting (**) $-$(*) gives:

$$2S_n-S_n=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}\right)$$

$$=1+\underbrace{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}-\frac{1}{2}-\frac{1}{2^2}-\frac{1}{2^3}-\ldots}_{=0}-\frac{1}{2^n}$$

$$S_n=1-\frac{1}{2^n} \qquad \left[\begin{array}{l}\text{Because subtracting ALL}\\ \text{middle terms gives 0}\end{array}\right]$$

Hence you are only left with the first and last terms of $2S_n - S_n$. Since $S_n = 1 - \frac{1}{2^n}$ therefore the sum of the infinite series, $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)$ is given by

$$\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) \qquad \left[\text{Substituting } S_n = 1 - \frac{1}{2^n}\right]$$

How do we find the value of $\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right)$ *?*

Because $\lim_{n\to\infty}\left(\frac{1}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$ by

(2.1) $\lim_{n\to\infty}(x^n) = 0$ if $|x| < 1$

Therefore

$$\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1 - \lim_{n\to\infty}\left(\frac{1}{2^n}\right) = 1 - 0 = 1$$

That is $\lim_{n\to\infty}(S_n) = 1$. We say the given series converges and its sum is 1 and this is written more formally as:

$$\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right) = 1$$

What does this, $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right) = 1$, *mean?*

Adding infinitely many terms of the form $\frac{1}{2^k}$ (and each of them positive) gives the sum equal to 1 or

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \ldots = 1$$

A2 **Geometric Series**

The above is an example of a geometric series. The general geometric series is given by $\sum_{n=1}^{\infty} ar^{n-1}$. *What are the first few terms of the series?*

$$\sum_{n=1}^{\infty} ar^{n-1} = ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + ar^{5-1} + \ldots + ar^{n-1} \qquad (a \neq 0)$$

$$= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

What do we mean by a geometric series?

An infinite series in which each term is obtained from the preceding term by multiplying by r. For example the second term, ar, is obtained from the first term, a, by multiplying it by a constant r. The symbol r is called the **common ratio** and a is called the **first term**. *In Example 1 what is* r and a *equal to?*

Common ratio $r = \frac{1}{2}$ and the first term $a = \frac{1}{2}$.

Next we test this series for convergence.

Example 2

Show that the geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$, converges for $|r|<1$ and diverges for $|r|\geq 1$.

Solution.

Let S_n be the nth partial sum of the geometric series. *How can we write* S_n *?*

S_n is the sum of the first n terms of the given series:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1}$$

Assume $r \neq 1$. Multiplying both sides by the common ratio r gives

$$\begin{aligned} rS_n &= ar + ar(r) + ar^2(r) + ar^3(r) + ar^4(r) + ... + ar^{n-1}(r) \\ &= ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^n \qquad [\text{Using the Rules of Indices}] \end{aligned}$$

Subtracting $rS_n - S_n$ gives

$$\begin{aligned} rS_n &= ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^n \\ -\ S_n &= -\left(a + ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1}\right) \end{aligned}$$

$$rS_n - S_n = ar^n - a \qquad [\text{Subtracting All other terms give } 0]$$

$$S_n(r-1) = a\left(r^n - 1\right) \qquad \begin{bmatrix} \text{Taking out Common Factors} \\ \text{from both sides} \end{bmatrix}$$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \qquad [\text{Dividing both sides by } r-1]$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r} \qquad \begin{bmatrix} \text{Multiplying Numerator} \\ \text{and Denominator by } -1 \end{bmatrix}$$

If $|r|<1$ then the sum of the geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$, is given by

$$\begin{aligned} \lim_{n\to\infty}(S_n) &= \lim_{n\to\infty}\left(\frac{a\left(1-r^n\right)}{1-r}\right) \qquad \left[\text{Substituting } S_n = \frac{a\left(1-r^n\right)}{1-r}\right] \\ &= \frac{a\left(1-\lim_{n\to\infty}\left(r^n\right)\right)}{1-r} \\ &= \frac{a(1-0)}{1-r} = \frac{a}{1-r} \end{aligned}$$

because $\lim_{n\to\infty}\left(r^n\right)=0$ by

(2.1) $\lim_{n\to\infty}\left(x^n\right)=0$ if $|x|<1$

Hence the geometric series converges for $|r|<1$. *Does the series* $\sum_{n=1}^{\infty} ar^{n-1}$ *converge for* $|r|\geq 1$ *?*

If $|r|\geq 1$ then $\lim_{n\to\infty}\left(r^n\right)$ diverges and therefore the sum of the geometric series

$$\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left(\frac{a\left(1-r^n\right)}{1-r}\right) \quad \text{diverges}$$

The geometric series converges for $|r|<1$ and the sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

but it diverges for $|r|\geq 1$.

Summarizing the above example we have that the geometric series is the infinite series defined as

(2.2) $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \quad (a \neq 0)$$

It is convergent if $|r|<1$ and the sum is

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{[\text{First Term}]}{1-[\text{Common Ratio}]}$$

If $|r|\geq 1$ then the geometric series diverges.

A3 **Examples of Geometric Series**

Example 3

Show that the following infinite series

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$$

converges and find its sum.

Solution

Writing out $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ we have

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \ldots$$

What type of series is this?

It is a geometric series because each term is $\frac{2}{3}$ the previous term. The common ratio $r=\frac{2}{3}$ and the first term $a=\frac{2}{3}$. *Does the series converge?*

Yes because the modulus of the common ratio, $|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$ so we can conclude by

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad \text{if } |r|<1$$

that $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges and the sum is

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2/3}{1-2/3}$$
$$= \frac{2/3}{1/3} = 2$$

This means adding the infinite number of terms of the form $\left(\frac{2}{3}\right)^n$ gives the sum equal to 2. That is

$$\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\ldots=2$$

Example 4

Show that the infinite series

$$\sum_{n=0}^{\infty} x^n \quad \text{where } |x|<1$$

converges and find its sum.

Solution

Is the given series a geometric series?

Writing out the series informally we have

$$\sum_{n=0}^{\infty} x^n = x^0+x^1+x^2+x^3+x^4+\ldots$$
$$=1+x+x^2+x^3+x^4+\ldots$$

Each term is obtained by multiplying the preceding term by x. Therefore yes it is a geometric series. *What is the first term,* a*, and the common ratio,* r*, equal to?*

First term $a=1$ and the common ratio $r=x$ and because $|r|=|x|<1$ therefore by

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad \text{if } |r|<1$$

the series $\sum_{n=0}^{\infty} x^n$ converges. The sum is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \left[=\frac{\text{First Term}}{1-(\text{Common Ratio})}\right]$$

provided $|x|<1$.

Note that the lower limit starts at $n=0$. First element of a series may not start at $n=1$ but $n=0$ or $n=10$ or $n=69$ and it is denoted by

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{n=10}^{\infty} a_n \text{ or } \sum_{n=69}^{\infty} a_n$$

Example 5

Determine whether the following series is convergent:

$$10-\frac{10}{3}+\frac{10}{9}-\frac{10}{27}+\ldots$$

If it is convergent then find its sum.

Solution

Is the given series a geometric series?

If we divide two consecutive terms then we have:

$$[\text{Second Term}] \div [\text{First Term}] = \left(-\frac{10}{3}\right) \div 10 = -\frac{1}{3} \qquad \text{or}$$

$$[\text{Third Term}] \div [\text{Second Term}] = \left(\frac{10}{9}\right) \div \left(-\frac{10}{3}\right) = -\frac{1}{3} \qquad \text{or}$$

$$[\text{Fourth Term}] \div [\text{Third Term}] = \left(-\frac{10}{27}\right) \div \left(\frac{10}{9}\right) = -\frac{1}{3}$$

Hence there is a common ratio, $r = -\frac{1}{3}$, between two consecutive terms. Normally it is easier to divide the first two terms of the series.

So we have a geometric series with a common ratio $r = -\frac{1}{3}$. Since

$$|r| = \left|-\frac{2}{3}\right| = \frac{2}{3} < 1$$

therefore by

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad \text{if } |r| < 1$$

the given series converges and the sum is equal to

$$10 - \frac{10}{3} + \frac{10}{9} - \frac{10}{27} + \ldots = \frac{10}{1 - \left(-\frac{1}{3}\right)} \qquad \left[= \frac{\text{First Term}}{1 - (\text{Commom Ratio})}\right]$$

$$= \frac{10}{4/3} = \frac{30}{4} = 7\frac{1}{2}$$

This means sum of the infinite series is $7\frac{1}{2}$, that is

$$10 - \frac{10}{3} + \frac{10}{9} - \frac{10}{27} + \ldots = 7\frac{1}{2}$$

Example 6

Determine whether the following series is convergent:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$$

If it is convergent then find its sum.

Solution

Writing out the first few terms of the series we have

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n} = \sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^n \qquad \left[\text{Because } \frac{(-3)^n}{2^n} = \left(-\frac{3}{2}\right)^n\right]$$

$$= -\frac{3}{2} + \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^4 + \ldots$$

Is this a geometric series?

Yes because each term is obtained by multiplying the preceding term by the common ratio $r=-\frac{3}{2}$. *Does the series converge?*
No because

$$|r|=\left|-\frac{3}{2}\right|=\frac{3}{2}\geq 1$$

therefore the series diverges. *What does this mean?*
Adding infinitely many terms

$$-\frac{3}{2}+\left(-\frac{3}{2}\right)^2+\left(-\frac{3}{2}\right)^3+\left(-\frac{3}{2}\right)^4+\ldots$$

does not approach a limit.

SUMMARY

An infinite series is denoted in compact form as $\sum_{n=1}^{\infty} a_n$ and can be informally written as:

$$a_1+a_2+a_3+a_4+\ldots$$

Geometric series is defined as $\sum_{n=1}^{\infty} ar^{n-1}$ where a is the first term and r is the common ratio. This series converges if $|r|<1$ and the sum is given by

(2.3) $$\sum_{n=1}^{\infty} ar^{n-1}=\frac{a}{1-r}=\frac{[\text{First Term}]}{1-[\text{Common Ratio}]}$$

But the series diverges for $|r|\geq 1$.