Exercise 7h

1. Write the following series in $\sum$ notation:

   a) $1+\sqrt{2}+\sqrt{3}+2+\ldots$ b) $2+4+6+8+\ldots$ c) $1+3+5+7+\ldots$

   * d) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ e) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$ f) $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\ldots$

2. Show that each of the following series is convergent and determine its sum:

   a) $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ b) $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ c) $\sum_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n$ d) $\sum_{n=1}^{\infty}\left(\frac{1}{m}\right)^n$

   where $m>1$.

3. Determine whether the following series is convergent. If it is convergent then find its sum.

   a) $\sum_{n=1}^{\infty}\left(\frac{1}{2^{2n-1}}\right)$ b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ c) $\sum_{n=1}^{\infty}(e)^n$ d) $\sum_{n=1}^{\infty}10\left(\frac{1}{3}\right)^n$

4. (Mechanics). A ball is dropped from a height of 10m and bounces to 55% of its previous height. Determine the total distance travelled by the ball.

5. (Mechanics). A hot air balloon rises 50m the first minute and then rises 65% of the previous minute for subsequent minutes. Determine the maximum rise of the balloon.

6. The Sierpinski triangle shown below is created by continually removing the middle of a triangle. (The shaded part shows the triangle that is removed).

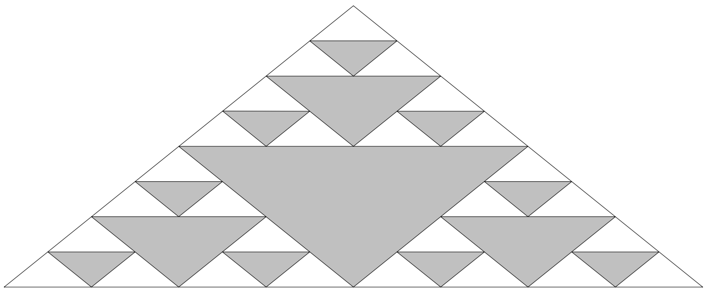

   Fig 38

   The area *A* removed is given by $A=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^n$. Find *A*.

7. Determine the sum *S* of the following infinite series:

   (a) $S=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$ (b) $S=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots$

   (c) $S=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

   What do you notice about your results?

8. A publishing company gives an academic a profit of £100 for the first year. For each subsequent year the profit falls by 9%. Determine the total possible profit.

9. Determine which of the following series converges. If it does converge then find its sum.

a) $8+4+2+1+...$ b) $3+6+12+24+...$ c) $16+12+9+\frac{27}{4}+...$

10. Show that the following series converges and find the sum in each case:

a) $\sum_{n=1}^{\infty}\frac{1}{x^n}$ where $|x|>1$ b) $\sum_{n=1}^{\infty}\left(\frac{x^n}{2^n}\right)$ where $|x|<2$

c) $\sum_{n=1}^{\infty}\frac{1}{(1+x)^n}$ where $x>0$ d) $\sum_{n=1}^{\infty}\frac{1}{(1+x^2)^n}$ where $x\neq 0$

Throughout the remaining questions, $\sum$ represents $\sum_{n=1}^{\infty}$.

11. Discuss convergence or divergence for each of the following series:

(a) $\sum\left(\frac{1}{(2n)!}\right)$ (b) $\sum_{n=0}^{\infty}\left(\frac{n!}{2^n}\right)$ (c) $\sum_{n=0}^{\infty}\left(\frac{n!}{3^n}\right)$ (d) $\sum\left(\frac{(n+1)^2}{2^n}\right)$

(e) $\sum\left(e^{-n}\right)$ (f) $\sum\left(\frac{n^2}{3^n}\right)$ (g) $\sum\left(\frac{10^n}{n!}\right)$ (h) $\sum\left(\frac{3^n n}{(n+1)^2}\right)$

(i) $\sum\left(\frac{n!}{(2n+1)!}\right)$ (j) $\sum\left(\frac{11^n}{2^{n+1}n}\right)$

12. Show that the ratio test fails for each of the following series:

(a) $\sum\left(\frac{1}{n^3}\right)$ (b) $\sum\left(\frac{1}{n+10}\right)$ (c) $\sum\left(\frac{1}{n^2+1}\right)$

13. * (a) Discuss convergence or divergence for each of the following series:

(i) $\sum\left(\frac{2^n n!}{n^n}\right)$ (ii) $\sum\left(\frac{3^n n!}{n^n}\right)$

(b) Determine the values of x for which the following series

$$\sum\left(\frac{x^n n!}{n^n}\right)$$

(i) converges (ii) diverges $\left(\text{Hint: } \lim_{n\to\infty}\left(\frac{n}{1+n}\right)^n = e\right)$

Solutions 7h

1. a) $\sum_{n=1}^{\infty}\sqrt{n}$ b) $\sum_{n=1}^{\infty}(2n)$ c) $\sum_{n=1}^{\infty}(2n-1)$ d) $\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\right]$

e) $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ f) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

2. a) $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=\frac{1}{2}$ b) $\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n=\frac{1}{3}$ c) $\sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n=\frac{1}{\pi-1}$ d) $\sum_{n=0}^{\infty}\left(\frac{1}{m}\right)^n=\frac{1}{m-1}$

3. a) Converges, $\sum_{n=1}^{\infty}\left(\frac{1}{2^{2n-1}}\right)=\frac{2}{3}$ b) Diverges c) Diverges

d) Converges, $\sum_{n=1}^{\infty}10\left(\frac{1}{3}\right)^n=5$

4. 22.22m (2dp)
5. 142.86m (2dp)
6. $A=1$ the whole area is removed.
7. (a) 1 (b) 1/3 (c) 1/9
8. £1111.11 (2dp)
9. a) Converges $\sum_{n=0}^{\infty}8\left(\frac{1}{2}\right)^n=16$ b) Diverges, $\sum_{n=0}^{\infty}3(2)^n$

c) Converges, $\sum_{n=0}^{\infty}16\left(\frac{3}{4}\right)^n=64$

10. a) $\sum_{n=1}^{\infty}\frac{1}{x^n}=\frac{1}{x-1}$ b) $\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n=\frac{x}{2-x}$ c) $\sum_{n=1}^{\infty}\frac{1}{(1+x)^n}=\frac{1}{x}$

d) $\sum_{n=1}^{\infty}\frac{1}{(1+x^2)^n}=\frac{1}{x^2}$

11. (a) Converges because $L=0$.
(b) Diverges because $L=+\infty$.
(c) Diverges because $L=+\infty$.
(d) Converges because $L=1/2$.
(e) Converges because $L=1/e$.
(f) Converges because $L=1/3$.
(g) Converges because $L=0$.
(h) Diverges because $L=3$.
(i) Converges because $L=0$.
(j) Diverges because $L=11/2$.

13. (a) (i) $L=\frac{2}{e}<1$ series converges (ii) Diverges because $L=\frac{3}{e}>1$.
(b) (i) $0<x<e$ (ii) $x>e$