Exercise 5(c)

Workbook questions in bold.

1. Find the number $N_0$ and the least positive integer $n$ such that $\forall n > N_0$ we have the inequality
$$\left| \frac{1}{2n} \right| < \varepsilon$$
for
(a) $\varepsilon = 0.1$
(b) $\varepsilon = 0.01$
(c) $\varepsilon = 1 \times 10^{-3}$
(d) $\varepsilon = 1 \times 10^{-6}$
Check your results.

2. Find the number $N_0$ and the least positive integer $n$ such that $\forall n > N_0$ we have the inequality
$$\left| \frac{2n + 1}{n + 1} - 2 \right| < \varepsilon$$
for
(a) $\varepsilon = 0.1$
(b) $\varepsilon = 0.01$
(c) $\varepsilon = 1 \times 10^{-3}$
(d) $\varepsilon = 1 \times 10^{-6}$

3. By using the formal definition of the limit of the sequence prove the following:
   (a) $\lim_{n \to \infty} \left( \frac{1}{n + 1} \right) = 0$
   (b) $\lim_{n \to \infty} \left( \frac{1}{n^2 + 1} \right) = 0$
   (c) $\lim_{n \to \infty} \left( \frac{1}{n^3 + 1} \right) = 0$

4. Prove that $\lim_{n \to \infty} \left( \frac{c}{n} \right) = 0$ for any $c \in \mathbb{R}$.

5. By using the formal definition of the limit of the sequence prove the following:
   (a) $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1$
   (b) $\lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) = 1$
   (c) $\lim_{n \to \infty} \left( 9 + \frac{1}{n} \right) = 9$
   (d) $\lim_{n \to \infty} \left( k + \frac{1}{n} \right) = k$ where $k$ is a real number

6. By using the formal definition of the limit of the sequence prove the following:
   (a) $\lim_{n \to \infty} \left( \frac{1}{\sqrt{n}} \right) = 0$
   (b) $\lim_{n \to \infty} \left( \frac{1}{\sqrt[n]{k}} \right) = 0$ where $k \in \mathbb{N}$
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7. Prove the following limits:
   
   (a) \( \lim_{n \to \infty} \left( \frac{n^2 - 1}{n^2 + 1} \right) = 1 \)
   
   (b) \( \lim_{n \to \infty} \left( \frac{n^2 - 1}{2^n} \right) = 0 \)

8. Prove that \( \lim_{n \to \infty} \left[ \frac{\cos(n)}{n} \right] = 0 \) using the formal definition of a limit of sequence.

9. (i) Prove the following \( \lim_{n \to \infty} \left( \frac{1}{2^n} \right) = 0 \)

   (ii) Show that if \( |x| > 1 \) then \( \lim_{n \to \infty} \left( \frac{1}{x^n} \right) = 0 \)

   (iii) Hence, or otherwise, prove that \( \lim_{n \to \infty} \left( e^{-n} \right) = 0 \)

10. By using the formal definition of the limit of the sequence prove that

    \( \lim_{n \to \infty} \left( \frac{1}{n^n} \right) = 0 \)

Solutions 5(c)

1. \( N_0 = \frac{1}{2\varepsilon} \).

   (a) \( n = 6 \)  \hspace{1cm}  (b) \( n = 51 \)  \hspace{1cm}  (c) \( n = 501 \).  \hspace{1cm}  (d) \( n = 500 \, 001 \)