Exercise 5b
1. Prove that the infimum of a set is unique.
2. Prove proposition (5.6).
3. Let \( S = \{ x \mid a \leq x \leq b \text{ and } x \in \mathbb{R} \} \). Prove that \( \inf (S) = a \).
4. A real number \( L \) of a non-empty subset, \( S \), of \( \mathbb{R} \) is the infimum of \( S \) \( \iff \)
   (i) For all \( s \in S \), \( s \geq L \)
   (ii) If \( m > L \) then there exists \( y \in S \) such that \( m > y \).
5. Let \( S \) and \( T \) be non-empty subsets of \( \mathbb{R} \) that are bounded below by real numbers. Let the set \( S + T \) be defined as
   \[
   S + T = \{ s + t \mid s \in S \text{ and } t \in T \}
   \]
   Prove that \( \inf (S + T) = \inf (S) + \inf (T) \).
6. Let \( S \) be a non-empty subset of \( \mathbb{R} \) which is bounded above and let \( k \in \mathbb{R} \). We define the set
   \[
   k + S = \{ k + s \mid s \in S \}
   \]
   Show that \( \sup (k + S) = k + \sup (S) \).
7. Let \( S \) be a non-empty subset of \( \mathbb{R} \) which is bounded above. Show that \( u \in \mathbb{R} \)
   is an upper bound of \( S \) if and only if \( t \in \mathbb{R} \) and \( t > u \Rightarrow t \notin S \).
8. Let \( S \) be a non-empty bounded set of real numbers and \( k \in \mathbb{R} \). We define the set
   \[
   kS = \{ ks \mid s \in S \}
   \]
   Prove the following results:
   (a) If \( k > 0 \) then
      (i) \( \sup (kS) = k \sup (S) \)
      (ii) \( \inf (kS) = k \inf (S) \)
   (b) If \( k < 0 \) then
      (i) \( \sup (kS) = k \inf (S) \)
      (ii) \( \inf (kS) = k \sup (S) \)
9. Let \( A \) and \( B \) be non-empty subsets of \( \mathbb{R} \) bounded above and below respectively. Prove that if for all \( a \in A \) and \( b \in B \)
   \[
   a \leq b
   \]
   then \( \sup (A) \leq \inf (B) \).
10. Let \( S \) and \( T \) be sets of positive numbers bounded above. Let the set \( ST \) be defined by
    \[
    ST = \{ st \mid s \in S \text{ and } t \in T \}
    \]
    Prove that \( \sup (ST) = \sup (S) \sup (T) \).
11. Let \( S \) and \( T \) be non-empty subsets of \( \mathbb{R} \) bounded above. Let the set \( ST \) be defined by
    \[
    ST = \{ st \mid s \in S \text{ and } t \in T \}
    \]
    Show that in general \( \sup (ST) \neq \sup (S) \sup (T) \).
Solutions 5b
1. Similar to the proof of (5.4) (i). Assume there are two \( L_1 \) and \( L_2 \) and show that they are equal, \( L_1 = L_2 \).
2. Similar to the proof of (5.5).
3. Similar to the proof of proposition (5.8). Suppose \( \inf(S) > a \) and \( \inf(S) < a \) and then arrive at a contradiction.
4. Apply proposition (5.6) with \( m = L + \varepsilon \) and an appropriate \( \varepsilon > 0 \).
5. Similar to the proof of proposition (5.9).
6. Use proof by contradiction with the supposition \( \sup(k + S) < k + \sup(S) \) and then use proposition (5.7) to produce the contradiction.
7. This is an if and only if proof so you need to go both ways, \( \Rightarrow \) and \( \Leftarrow \).
8. Use proof by contradiction in each case. Apply proposition (5.7) or the proposition in Question ?? to produce the contradiction.
9. *Proof.* Suppose \( \sup(A) > \inf(B) \). By the proposition in Question ??, \( \exists b \in B \) such that \( \sup(A) > b \). Since \( b < \sup(A) \), by proposition (5.7) \( \exists a \in A \) such that \( b < a \). This contradicts \( a \leq b \). Hence \( \sup(A) \leq \inf(B) \).
10. Again use proof by contradiction and proposition (5.7).
11. Consider the sets:
\[
S = \{ s \mid s < -1 \text{ and } s \in \mathbb{R} \} \\
T = \{ t \mid t < -2 \text{ and } t \in \mathbb{R} \}
\]
Then \( ST = \{ st \mid st > 2 \} \). Hence the set \( ST \) is not bounded above and so cannot have a supremum.