Exercise 3(e)

Workbook questions in bold, Questions 4 and 5.

1. Let $A$ be any set and $I_A$ be the identity function on the set $A$.
   (i) Prove that $I_A$ is bijective.
   (ii) Determine $I_A^{-1}$.

2. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x-1)^3 + 2$.
   (i) Show that the given function $f$ is bijective.
   (ii) Determine the formula for $f^{-1}(x)$.
   (iii) Determine $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

3. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let $f: A \to B$ be defined by $f(1) = b$, $f(2) = c$, $f(3) = a$.
   (i) Show that the function $f$ is bijective.
   (ii) Specify $f^{-1}$.
   (iii) Find $f \circ f^{-1}$ and $f^{-1} \circ f$.

4. Let $f: A \to B$ be a bijective function. Prove that $f \circ f^{-1} = I_B$.

5. Let $g: A \to B$ and $f: B \to C$ be injective functions. Prove that $f \circ g: A \to C$ is also injective.

6. Let $g: A \to B$ and $f: B \to C$ be surjective functions. Prove that $f \circ g: A \to C$ is also surjective.

7. Let $g: A \to B$ and $f: B \to C$ be bijective functions. Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

8. Let $f: A \to B$ be a function. Prove that there exists a function $g: B \to A$ such that $g \circ f = I_A$ if and only if $f$ is an injection.
   The function $g$ is called the left inverse of $f$.

9. Let $f: A \to B$ be a function. Prove that there exists a function $g: B \to A$ such that $f \circ g = I_B$ if and only if $f$ is a surjection.
   The function $g$ is called the right inverse of $f$. 