Exercise 3(a)

1. Let \( A = \{1, 2, 3, 4\} \) and \( B = \{a, b, c, d\} \). Let \( f : A \rightarrow B \) be defined by the following diagrams. In each case decide whether \( f \) is a function or not. If \( f \) is not a function then explain why not.

   (a) 
   ![Diagram](image1)

   (b) 
   ![Diagram](image2)

   (c) 
   ![Diagram](image3)

   (d) 
   ![Diagram](image4)

Fig 11

2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be given by 
\[
f(x) = x^2 + x
\]
Find the images of the following under the function \( f \).

   (a) 2  
   (b) 7  
   (c) \( y \)  
   (d) \( a + b \)
3. Consider Dirichlet’s function

\[ f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases} \]

Determine the images of the following:
(a) \( \pi \)  
(b) \( e \)  
(c) \( 1 \)  
(d) \( \sqrt{3} \)  
(e) \( \frac{2}{3} \)  
(f) \( \sqrt{4} \)  
(g) \( 1 + \sqrt{5} \)  
(h) \( 1.414 \)

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be given:

\[ f(x) = \sqrt{x} \]

The same as in Example 6. Redefine the codomain so that this is a function.

5. Let \( f : \mathbb{N}/\{1\} \to \mathbb{Z} \) be given by

\[ f(n) = \begin{cases} n & \text{if } n \text{ is prime} \\ -n & \text{if } n \text{ is composite} \end{cases} \]

Determine the following:
(a) \( f(13) \)  
(b) \( f(33) \)  
(c) \( f\left(\frac{1}{2}\right) \)

What is the range of \( f \)?
(Remember a composite number is a positive integer that is not prime).

6. Determine the domain, codomain and range for each of the following functions:
Let \( A = \{-3, -1, 0, 1, 3\} \) and \( f : A \to \mathbb{R} \) be given by
(a) \( f(x) = x^2 - 1 \)  
(b) \( f(x) = x^2 + 1 \)  
(c) \( f(x) = x^2 + 5x - 2 \)

7. Determine the range of the following functions.

Let \( A = \left\{ -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2} \right\} \) and \( g : A \to \mathbb{R} \) be given by
(a) \( g(x) = \cos(x) \)  
(b) \( g(x) = \sin(x) \)  
(c) \( g(x) = \cos^2(x) + \sin^2(x) \)

8. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function given by \( f(x) = x^2 + 3x - 2 \). Determine
(a) \( f(y) \)  
(b) \( f\left(\sqrt{2}\right) \)  
(c) \( f(a + b) \)

9. Let \( g : \mathbb{R}^+ \to \mathbb{R} \). Determine \( g(x + y) \) for each of the following functions:
(a) \( g(x) = e^x \)  
(b) \( g(x) = \ln(x) \)  
(c) \( \sin(x) \)  
(d) \( \cos(x) \)

10. The factorial function \( f : \mathbb{N} \to \mathbb{N} \) is defined by

\[ f(n) = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n \]

Determine \( f(5) \), \( f(7) \), \( f(10) \), \( f(69) \) and \( f\left(\frac{1}{2}\right) \).
11. The tau function $\tau : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\tau(n) = \text{The number of positive divisors of } n$$

For example $\tau(6) = 4$ because the following 4 positive integers - 1, 2, 3 and 6 divide into 6 exactly.
Determine $\tau(10)$, $\tau(12)$, $\tau(20)$, $\tau(25)$ and $\tau(40)$.

12. The signum function $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\text{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Determine the domain, codomain and range of $\text{sgn}(x)$.

13. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be given by $g(n) = 2^n$. Determine the range of $g$. Show that

(i) $g(n + 1) = 2g(n)$
(ii) $g(n + 2) = 4g(n)$
(iii) $g(n + 3) = 8g(n)$

Prove that $g(n + m) = 2^m g(n)$.

14. Let $g : \mathbb{Z} \rightarrow \mathbb{Q}$ be given by $g(n) = 2^n$. Determine the range of $g$. Prove that $g(n + m) = 2^m g(n)$

Some Solutions to Exercise 3a

4. Redefine the codomain to be the set of complex numbers, $\mathbb{C}$. Hence we have the function $g : \mathbb{R} \rightarrow \mathbb{C}$ given by $g(x) = \sqrt{x}$.

5. (a) $f(13) = 13$  
(b) $f(33) = -33$  
(c) $f\left(\frac{1}{2}\right)$ is not defined because $\frac{1}{2}$ is not in the domain. The domain is all the natural numbers apart from 1. The range is all the integers apart from 0, 1 and $-1$. Hence this can be written as $\mathbb{Z}/\{-1, 0, 1\}$.