

Exercise 2e

Throughout this exercise \sum represents $\sum_{n=1}^{\infty}$.

1. Discuss convergence or divergence for each of the following series:

- (a) $\sum \left(\frac{1}{(2n)!} \right)$ (b) $\sum_{n=0}^{\infty} \left(\frac{n!}{2^n} \right)$ (c) $\sum_{n=0}^{\infty} \left(\frac{n!}{3^n} \right)$
(d) $\sum \left(\frac{(n+1)^2}{2^n} \right)$ (e) $\sum (e^{-n})$ (f) $\sum \left(\frac{n^2}{3^n} \right)$
(g) $\sum \left(\frac{10^n}{n!} \right)$ (h) $\sum \left(\frac{(n!)^2}{(2n)!} \right)$ (i) $\sum \left(\frac{(\sqrt{7}-1)^n}{n^2+1} \right)$
(j) $\sum \left(\frac{3^n n}{(n+1)^2} \right)$ (k) $\sum \left(\frac{n!}{(2n+1)!} \right)$ (m) $\sum \left(\frac{5^n}{2^{n+1}n} \right)$
(n) $\sum (n^{-n})$ (o) $\sum (n^n e^{-n})$

2. (a) Discuss convergence or divergence for each of the following series:

- (i) $\sum \left(\frac{2^n n!}{n^n} \right)$ (ii) $\sum \left(\frac{3^n n!}{n^n} \right)$

(b) Determine the values of x for which the following series

$$\sum \left(\frac{x^n n!}{n^n} \right)$$

- (i) converges (ii) diverges

(Hint: $\lim_{n \rightarrow \infty} \left(\frac{n}{1+n} \right)^n = e$)

3. Test the following series for convergence $\sum \left(\frac{n^n}{n!} \right)$.

4. (i) Show that the ratio test fails for each of the following series:

- (a) $\sum \left(\frac{1}{n^3} \right)$ (b) $\sum \left(\frac{1}{n+1} \right)$ (c) $\sum \left(\frac{1}{n^2+1} \right)$
(d) $\sum \left(\frac{1}{\sqrt{n^2+1}} \right)$ (e) $\sum \left(\frac{1}{\sqrt{n+1}} \right)$ (f) $\sum \left(\frac{1}{n^p} \right)$

(ii) Test each of the series in part (a).

Solutions

1. (a) Converges because $L = 0$.
(a) Diverges because $L = +\infty$.
(b) Diverges because $L = +\infty$.
(c) Converges because $L = 1/2$.

- (d) Converges because $L = 1/e$.
 - (e) Converges because $L = 1/3$.
 - (f) Converges because $L = 0$.
 - (g) Converges because $L = 1/4$.
 - (h) Diverges because $L = \sqrt{7} - 1$.
 - (i) Diverges because $L = 3$.
 - (j) Converges because $L = 0$.
 - (k) Converges because $L = 2/3$.
 - (l) Diverges because $L = 5/2$.
 - (m) Converges because $L = 0$.
 - (n) Diverges because $L = +\infty$.
2. (a) (i) $L = \frac{2}{e} < 1$ series converges (ii) Diverges because $L = \frac{3}{e} > 1$.
- (b) (i) $0 < x < e$ (ii) $x > e$
3. Diverges because $L = e > 1$.
4. (i) In each case $L = 1$.
- (ii) (a) Converges because $p = 3 > 1$.
- (b) Diverges. Use the limit comparison test and compare with $1/n$.
- (c) Converges. Use the normal comparison test with $1/n^2$.
- (d) Diverges. Use the limit comparison test and compare with $1/n$.
- (e) Diverges. Use the limit comparison test and compare with $1/\sqrt{n}$.
- (f) This is the p-series test with convergence for $p > 1$ and divergence for $p \leq 1$.