Exercise 1(h)

1. Prove that for all natural numbers $n$, $9$ divides $10^n - 1$.

2. Prove that for all natural numbers $n$, 
   \[ 3 \mid (n^3 - n) \].

3. Show that for every natural number $n$ 
   \[ 3 \mid n(n+1)(n+2) \]

4. Prove that for all natural numbers $n$ 
   \[ n^2 - n \] is an even number.

5. Prove that for all natural numbers $n$ 
   \[ a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \] \[ r \neq 1 \] 
   where $a$ and $r$ are real numbers. 
   [This is a geometric series with first term equal to $a$ and common ratio $r$]

6. Prove that for all natural numbers $n$ we have the following trigonometric identity: 
   \[ \sin(x) + \sin(2x) + \ldots + \sin(nx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \] 
   where $x$ is a real number such that $\sin\left(\frac{x}{2}\right) \neq 0$.

7. Prove the binomial theorem for the natural number $n$:
   If $a$ and $b$ are real numbers then the binomial theorem says that for all natural numbers $n$ 
   \[ (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n \]