Exercise 1(d)

Throughout this exercise lower case letters such as \(a, b, c,...,n,...\) represent integers.

1. Prove the following propositions:
   (a) If \(n\) and \(m\) are even then their sum \(n + m\) is even.
   (b) If \(n\) and \(m\) are even then their subtraction \(n - m\) is even.
   (c) If \(n\) and \(m\) are odd then their subtraction \(n - m\) is even.
   (d) If \(n\) is an odd number then \(n^2\) is also odd.
   (e) If \(n\) is even and \(m\) is odd then their sum \(m + n\) is odd.
   (f) If \(n\) is odd and \(m\) is odd then their product \(nm\) is odd.
   (g) If \(n\) is any integer and \(m\) is even then their product \(nm\) is even.

2. Prove the following propositions:
   (i) \(n\) is odd \(\Rightarrow\) \(n + 1\) is even
   (ii) For any integer \(n\) we have \(n(n + 1)\) is even
       [Hint: Use proposition (1.4)].

3. Prove that if \(n\) is odd then \(n^3 - 1\) is even.
   [Hint: Use the propositions proved in Question 1].

4. Prove the following propositions:
   (a) \(a | 0\)       (b) \(a | a\)        (c) \(1 | a\)    (d) \(a | a^2\)
   (e) \(a | a^n\) where \(n \geq 1\) is an integer
   (f) \(a | b\) and \(a | c\) \(\Rightarrow\) \(a | (b + c)\)
   (g) \(a | b\) and \(a | c\) \(\Rightarrow\) \(a^2 | bc\)
   (h) \(ac | bc\) \(\Rightarrow\) \(a | b\) where \(c \neq 0\)
   (i) \(a | b\) and \(c | d\) \(\Rightarrow\) \(ac | bd\)

The remaining three questions are more difficult.

5. Prove the following proposition:
   If \(n\) is odd then
   (a) \(8 | (n^2 - 1)\)
   (b) \(32 | (n^2 + 3)(n^2 + 7)\)

6. Show that if the last digit of an integer \(n\) is even then \(n\) is even.
   [Hint: Write \(n\) as
   \[n = a_m a_{m-1} a_{m-2}...a_2 a_1 a_0 \quad (m > 1)\]
   where \(a_m, a_{m-1}, a_{m-2},..., a_2, a_1, a_0\) are the digits of \(n\). Note that \(a_0\) is the last digit. Hence \(n\) is written as
   \[n = (a_m \times 10^m) + (a_{m-1} \times 10^{m-1}) + (a_{m-2} \times 10^{m-2}) + ... + (a_2 \times 10^2) + (a_1 \times 10^1) + a_0\]

7. Show that if the last digit of an integer \(n\) is odd then \(n\) is odd.