1. (a) We are given the series $\sum_{n=0}^{\infty}\left(x^n\right)$. We use the ratio test (7.31) with

$$a_n = x^n \text{ and the n+1 term is } a_{n+1} = x^{n+1}$$

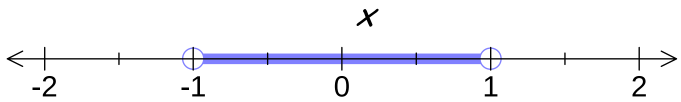
We have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| = |x|$$

The series converges if $L<1$ so in our case $|x|<1$. *What happens at* $x=1$*?*

Substituting $x=1$ into the given power series $\sum_{n=0}^{\infty}\left(x^n\right)$ gives $\sum_{n=0}^{\infty}\left(1^n\right)$. We have

$\lim_{n\to\infty}\left(1^n\right)\neq 0$ therefore $\sum_{n=0}^{\infty}\left(1^n\right)$ diverges. Similarly the series diverges at $x=-1$.



Hence the interval of convergence is $-1<x<1$ and the radius of convergence $R=1$.

(b) We have to find the interval and radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{x^n}{2n}\right)$. Again we use the ratio test with $a_n = \frac{x^n}{2n}$ which implies that $a_{n+1} = \frac{x^{n+1}}{2(n+1)}$.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2(n+1)} \div \frac{x^n}{2n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{2(n+1)} \times \frac{2n}{x^n}\right| \qquad \left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$\underbrace{=}_{\text{Cancelling like terms}} |x|\lim_{n\to\infty}\left(\frac{2n}{2n+2}\right) \underbrace{=}_{\text{Dividing numerator and denominator by } n} |x|\underbrace{\lim_{n\to\infty}\left(\frac{2}{2+2/n}\right)}_{=1} = |x|$$

As part (a) the series converges for $|x|<1$. At $x=1$ we have

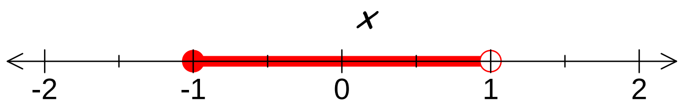
$$\sum_{n=1}^{\infty}\left(\frac{1^n}{2n}\right) \text{ which diverges by (7.30) (a)}$$

At $x=-1$ we have $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{2n}\right)$ which converges by (7.30) (b). Hence the interval of convergence is $-1\le x<1$ and radius of convergence is $R=1$.

(7.30) (a) $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ converges for $p>1$ (p is **greater than** 1)

(7.30) (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right)$ converges for $p\ge 1$ (p is **greater than or equal to** 1)

We can represent this convergence as:



(c) Similarly using the ratio test with $a_n = \dfrac{x^n}{n^2}$ and $a_{n+1} = \dfrac{x^{n+1}}{(n+1)^2}$ we have

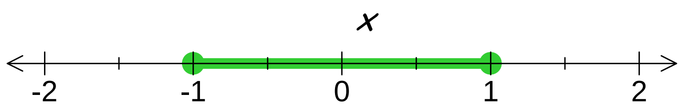
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\times\frac{n^2}{x^n}\right|$$

$$= \lim_{n\to\infty}|x|\left(\frac{n}{n+1}\right)^2 \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}} |x|\underbrace{\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2}_{=1} = |x|$$

The given power series converges for $|x|<1$. At $x=1$ and $x=-1$ we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \text{ respectively.}$$

Both these series converge by the *p* test because $p=2>1$. Our interval of



convergence is $-1\le x\le 1$ and the radius of convergence is $R=1$.

(d) Let $a_n = \dfrac{nx^n}{2n+1}$ and then $a_{n+1} = \dfrac{(n+1)x^{n+1}}{2(n+1)+1} = \dfrac{(n+1)x^{n+1}}{2n+3}$. Using the ratio test with these gives

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{2n+3}\div\frac{nx^n}{2n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{2n+3}\times\frac{(2n+1)}{nx^n}\right|$$

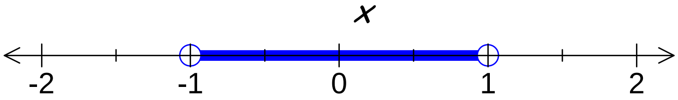
$$= |x|\lim_{n\to\infty}\left(\frac{(n+1)}{(2n+3)}\times\frac{(2n+1)}{n}\right)$$

$$\underbrace{=}_{\text{Expanding}} |x|\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{2n^2+3n}\right) \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n^2}} |x|\underbrace{\lim_{n\to\infty}\left(\frac{2+3/n+1/n^2}{2+3/n}\right)}_{=1} = |x|$$

The given power series converges for $|x|<1$. At $x=1$ and $x=-1$ we have

$$\sum_{n=1}^{\infty}\frac{n}{2n+1} \text{ and } \sum_{n=1}^{\infty}\frac{(-1)^n n}{2n+1} \text{ respectively.}$$

Both these series diverge because the nth term does not tend to zero as $n\to\infty$. Hence the interval of convergence is $-1<x<1$ and $R=1$. Illustrating this:



(e) We are given the series $\sum_{n=0}^{\infty}\left(\frac{x}{\sqrt{2}}\right)^n$. Using the ratio test with

$$a_n = \left(\frac{x}{\sqrt{2}}\right)^n \text{ and } a_{n+1} = \left(\frac{x}{\sqrt{2}}\right)^{n+1}$$

gives

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{x}{\sqrt{2}}\right)^{n+1} \div \left(\frac{x}{\sqrt{2}}\right)^n\right|$$

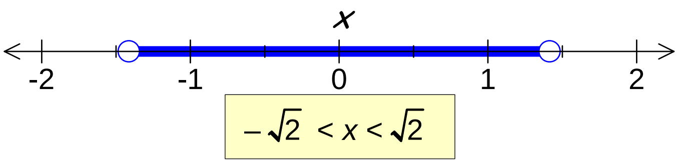
$$= \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{\left(\sqrt{2}\right)^{n+1}}\right)\times\frac{\left(\sqrt{2}\right)^n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{\sqrt{2}}\right| = \frac{|x|}{\sqrt{2}}$$

Remember the ratio test says that the series converges for $L<1$ which in this case means that $\frac{|x|}{\sqrt{2}}<1 \quad \Leftrightarrow \quad |x|<\sqrt{2}$. *What about when* $x=\sqrt{2}$ *?*

The series becomes $\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty}(1)^n$ which diverges because the nth term does **not** tend to zero. Similarly at $x=-\sqrt{2}$ we have $\sum_{n=0}^{\infty}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty}(-1)^n$ which also diverges.

The interval of convergence is $-\sqrt{2}<x<\sqrt{2}$:



Clearly the radius of convergence $R=\sqrt{2}$.

(f) *What is* a_n *and* a_{n+1} *equal to for the series* $\sum_{n=0}^{\infty}\left(\frac{n^2}{2^n}\right)x^n$ *?*

$$a_n = \left(\frac{n^2}{2^n}\right)x^n \text{ and } a_{n+1} = \left(\frac{(n+1)^2}{2^{n+1}}\right)x^{n+1}$$

We have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)^2}{2^{n+1}}\right)x^{n+1}\times\left(\frac{2^n}{n^2x^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\left(\frac{(n+1)^2}{n^2}\right)\right| \qquad \text{[Cancelling like terms]}$$

$$= \frac{|x|}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{|x|}{2}\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2}_{=1} = \frac{|x|}{2} \qquad \left[\text{Because } \frac{n+1}{n} = \frac{n}{n}+\frac{1}{n} = 1+\frac{1}{n}\right]$$

The series converges for L less than 1 which in this case means that

$$\frac{|x|}{2} < 1$$

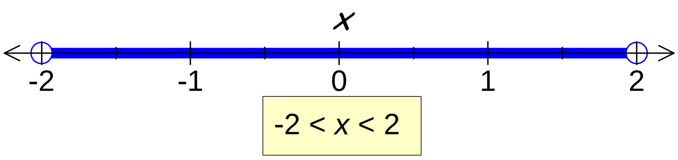
Hence the series converges for $|x| < 2$. Remember the ratio test fails for $L = 1$. At $L = 1$ we have two values of x which are $x = 2$ and $x = -2$. At $x = 2$ we have

$$\sum_{n=0}^{\infty}\left(\frac{n^2}{2^n}\right)2^n = \sum_{n=0}^{\infty}n^2$$

which diverges because the nth term does **not** tend to zero. Similarly at $x = -2$:

$$\sum_{n=0}^{\infty}\left(\frac{n^2}{2^n}\right)(-2)^n = \sum_{n=0}^{\infty}(-1)^n n^2$$

Again this diverges. The interval of convergence is $-2 < x < 2$ which is illustrated:



The radius of convergence $R = 2$.

---

2. (a) We are given the series $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$. *What is* a_n *and* a_{n+1} *equal to in this case?*

$$a_n = \frac{x^n}{n!} \text{ and } a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

Using the ratio test we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\div\frac{x^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}\right| \qquad \left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$= |x|\lim_{n\to\infty}\left(\frac{n!}{(n+1)!}\right) = |x|\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = |x|(0) = 0$$

The series converges for all real numbers x. The interval of convergence is $-\infty < x < +\infty$.

(b) Similarly for $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}x^n}{n}\right)$ we have $a_n=\frac{(-1)^{n-1}x^n}{n}$ and replacing n with $n+1$ gives $a_{n+1}=\frac{(-1)^{n+1-1}x^{n+1}}{n+1}=\frac{(-1)^n x^{n+1}}{n+1}$. Applying the ratio test we have

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{n+1}\times\frac{n}{(-1)^{n-1}x^n}\right|$$

$$=\lim_{n\to\infty}\left|(-1)|x|\frac{n}{n+1}\right|$$

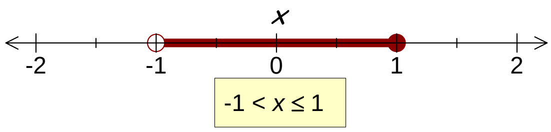
$$=|x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}|x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)=|x|$$

The series converges for $|x|<1$. At $x=1$ we have $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}1^n}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n}\right)$ which is an alternating series and converges because of the p test with $p=1$.

With $x=-1$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}(-1)^n}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{(-1)^{2n-1}}{n}\right)=\sum_{n=1}^{\infty}\left((-1)\frac{(-1)^{2n}}{n}\right)=\sum_{n=1}^{\infty}\left((-1)\frac{1}{n}\right)$$

This is the harmonic series with minus sign in front so it diverges. Hence the interval of convergence is $-1<x\le 1$.



(c) We are given $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$ so let $a_n=\frac{(-1)^n x^{2n}}{(2n)!}$ then

$$a_{n+1}=\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}=\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}$$

Substituting these into $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\times\frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$=\lim_{n\to\infty}\left|(-1)x^2\frac{(2n)!}{(2n+2)!}\right| \qquad \text{[Cancelling like terms]}$$

$$=x^2\underbrace{\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+1)}\right|}_{=0}=x^2(0)=0$$

The given series converges for **all** real numbers so interval of convergence is $-\infty<x<+\infty$.

(d) *What are nth and (n+1)th terms for the series* $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)}\right)$*?*

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)} \text{ and } a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)}$$

Applying the ratio test for convergence we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \div \frac{(-1)^n x^{2n+1}}{(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \times \frac{(2n+1)}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|(-1)x^2 \frac{(2n+1)}{(2n+3)}\right| \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}} x^2 \lim_{n\to\infty}\left(\frac{2+1/n}{2+3/n}\right) = x^2\left[\frac{2+0}{2+0}\right] = x^2$$

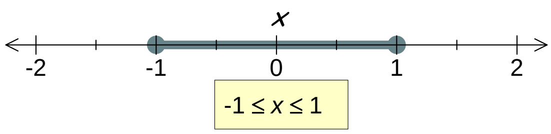
The given series converges for $L = x^2 < 1$. *What happens at* $x = 1$*?*
Substituting $x = 1$ into the given series we have

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n 1^{2n+1}}{(2n+1)}\right) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)$$

This is an alternating series and converges by the p-test with $p = 1$. At $x = -1$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n (-1)^{2n+1}}{(2n+1)}\right) = \sum_{n=0}^{\infty}\left(\frac{(-1)^{3n+1}}{2n+1}\right)$$

Again this is an alternating series and it converges by the p-test.



Combining these we have that the interval of convergence is $-1 \le x \le 1$.

(e) We are given the series $\sum_{n=0}^{\infty}\left(x^{2n}\right)$. For this

$$a_n = x^{2n} \text{ and the next term is } a_{n+1} = x^{2(n+1)} = x^{2n+2}$$

Substituting these into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\right| = \lim_{n\to\infty}\left|x^2\right| = x^2$$

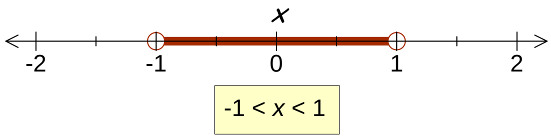
For what values of x does the series converge?

$$x^2 < 1$$

This means the series converges for $-1 < x < 1$. Remember the ratio test fails for $L = 1$.
At $x = 1$ and $x = -1$ we have

$$\sum_{n=0}^{\infty}\left(1^{2n}\right) \text{ and } \sum_{n=0}^{\infty}(-1)^{2n} \text{ respectively.}$$

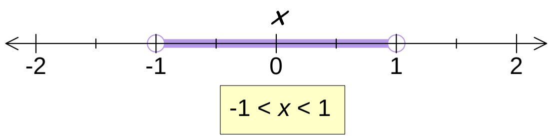
In both these cases the series diverges because the nth term does **not** go to zero as $n \to \infty$. The interval of convergence is $-1 < x < 1$ which can be illustrated as:



(f) In this case $\sum_{n=0}^{\infty}\left(x^{n^2}\right)$ we let $a_n = x^{n^2}$ and then $a_{n+1} = x^{(n+1)^2} = x^{n^2+2n+1}$. Substituting these into the ratio test yields

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n^2+2n+1}}{x^{n^2}}\right| \underbrace{=}_{\text{Using the rules of indices}} \lim_{n\to\infty}\left|x^{2n+1}\right|$$

The series converges for $L = \lim_{n\to\infty}\left|x^{2n+1}\right| < 1$ which means that $-1 < x < 1$. At $x = \pm 1$ we have $\sum_{n=0}^{\infty}(\pm 1)^{n^2}$. Because $\lim_{n\to\infty}(\pm 1)^{n^2} \neq 0$ therefore the series diverges at $x = \pm 1$.



The interval of convergence is $-1 < x < 1$.

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3. In each of these cases the series is expanded about a non-zero point.

(a) We are given the series $\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{n}\right)$. *What is a_n and a_{n+1} equal to in this case?*

Let $a_n = \frac{(x+1)^n}{n}$ then $a_{n+1} = \frac{(x+1)^{n+1}}{n+1}$. Substituting these into the ratio test yields:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{n+1} \div \frac{(x+1)^n}{n}\right|$$

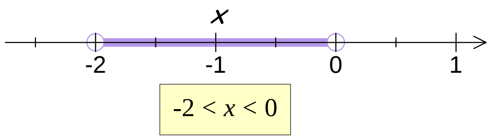
$$= \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{n+1} \times \frac{n}{(x+1)^n}\right| \qquad \left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$= \lim_{n\to\infty}\left|(x+1) \times \frac{n}{n+1}\right| \qquad [\text{Cancelling like terms}]$$

$$= |x+1|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}} |x+1|\underbrace{\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)}_{=1} = |x+1|$$

The series converges for $|x+1| < 1$. This modulus means that x is at most 1 away from the centre -1 because we can write this inequality as $|x-(-1)| < 1$:

This is **not** the interval of convergence.



We need to test the two end points. At $x = 0$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{(0+1)^n}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$$

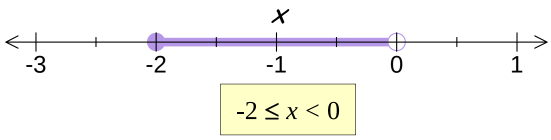
This series diverges because it is the harmonic series, see page 377.
The other end point is at $x = -2$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{(-2+1)^n}{n}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

This is the alternating harmonic series which does converge because of the p-test:

(7.30) (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right)$ converges for $p \ge 1$ (p is **greater than or equal to** 1)

The interval of convergence is $-2 \le x < 0$ which we can illustrate as follows:



(b) *How do we find for what values of x does the series* $\sum_{n=1}^{\infty}\left(\frac{(x-2)^n}{n^2}\right)$ *converge?*

By applying the ratio test with $a_n = \frac{(x-2)^n}{n^2}$. *What is the next term* a_{n+1} *equal to?*

$$a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2}$$

Substituting these, $a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2}$ and $a_n = \frac{(x-2)^n}{n^2}$, into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

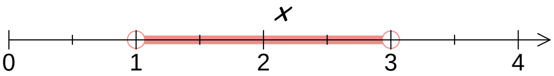
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2} \div \frac{(x-2)^n}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2} \times \frac{n^2}{(x-2)^n}\right| \qquad \left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$= |x-2|\lim_{n\to\infty}\frac{n^2}{(n+1)^2}$$

$$= |x-2|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}} |x-2|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2 = |x-2|$$

The series converges for $|x-2| < 1$ which we can illustrate as (this is not the interval of convergence):



$1 < x < 3$

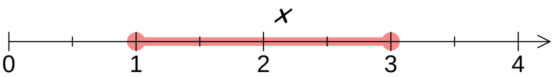
We need to check the two end points, $x = 1$ and $x = 3$. At $x = 1$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(x-2)^n}{n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(1-2)^n}{n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)$$

This series converges by the p-test because in this case $p = 2$ which is greater than 1.
At the other end point $x = 3$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(x-2)^n}{n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{(3-2)^n}{n^2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

Again this series converges by the p-test. Hence the interval of convergence is



$1 \le x \le 3$

(c) In this case we have $\sum_{n=0}^{\infty}\left(\frac{(-1)^n (x-1)^n}{2^n}\right)$ and so $a_n = \frac{(-1)^n (x-1)^n}{2^n}$. The next term is:

$$a_{n+1} = \frac{(-1)^{n+1}(x-1)^{n+1}}{2^{n+1}}$$

Substituting these into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$:

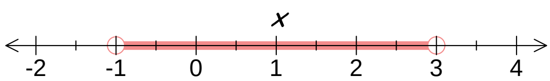
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{2^{n+1}} \div \frac{(-1)^n (x-1)^n}{2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{2^{n+1}} \times \frac{2^n}{(-1)^n (x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(x-1)}{2}\right| \qquad \text{[Cancelling like terms]}$$

$$= \left|\frac{x-1}{2}\right|$$

The series converges for $\left|\frac{x-1}{2}\right| = \frac{|x-1|}{2} < 1$ and this implies $|x-1| < 2$. *What does inequality mean?*
This means that *x* lies within 2 units from the centre 1. We have



$-1 < x < 3$

We need to test two end points for convergence. *What is the series at* $x = 3$ *?*

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n (x-1)^n}{2^n}\right)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n (3-1)^n}{2^n}\right)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n 2^n}{2^n}\right)=\sum_{n=0}^{\infty}(-1)^n$$

Since the nth term does **not** tend to zero as $n\to\infty$ so this series diverges. Similarly at the other end point $x=-1$ we have

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n (-1-1)^n}{2^n}\right)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n (-2)^n}{2^n}\right)=\sum_{n=0}^{\infty}\left((-1)^n (-1)^n\right)=\sum_{n=0}^{\infty}(-1)^{2n}$$

Again this series diverges.
Hence the interval of convergence is $-1<x<3$.

(d) This is very similar to the above parts. We are given the series $\sum_{n=0}^{\infty}\left(\frac{n}{2n+1}\left(\frac{x+3}{2}\right)^n\right)$.

Let $a_n=\frac{n}{2n+1}\left(\frac{x+3}{2}\right)^n$ then $a_{n+1}=\frac{n+1}{2(n+1)+1}\left(\frac{x+3}{2}\right)^{n+1}=\frac{n+1}{2n+3}\left(\frac{x+3}{2}\right)^{n+1}$. Applying the ratio test we have

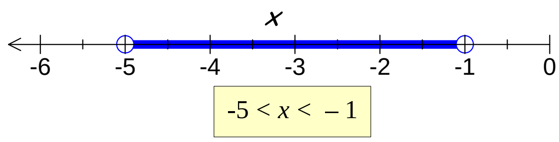
$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{n+1}{2n+3}\left(\frac{x+3}{2}\right)^{n+1}\div\frac{n}{2n+1}\left(\frac{x+3}{2}\right)^{n}\right|$$

$$=\frac{|x+3|}{2}\lim_{n\to\infty}\left(\frac{(n+1)}{2n+3}\times\frac{(2n+1)}{n}\right)\qquad\left[\begin{array}{l}\text{Applying the rules}\\ \text{of indices }\ b^m\div b^n=b^{m-n}\end{array}\right]$$

$$=\frac{|x+3|}{2}\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{2n^2+3n}\right)$$

$$\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n^2}}\frac{|x+3|}{2}\underbrace{\lim_{n\to\infty}\left(\frac{2+3/n+1/n^2}{2+3/n}\right)}_{=1}=\frac{|x+3|}{2}$$

The series converges for $\frac{|x+3|}{2}<1$ which is equivalent to $|x+3|<2$. This means that x lies within 2 units of the centre -3 because we can write the inequality as $|x-(-3)|<2$:



At the two end points $x=-5$ and $x=-1$ the series diverges because in both cases the nth term does **not** tend to zero as $n\to\infty$. Hence the interval of convergence is $-5<x<-1$.

---

4. We need to show that $L=1$ for each of the given series. $L=1$ means that the ratio test fails.

(a) For the given series $\sum\left(\frac{1}{n}\right)$ we let $a_n=\frac{1}{n}$ therefore $a_{n+1}=\frac{1}{n+1}$. The ratio L is:

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\div\frac{1}{n}\right|$$

$$=\lim_{n\to\infty}\left(\frac{1}{n+1}\times\frac{n}{1}\right)\qquad\left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)=\frac{1}{1+0}=1$$

Hence the ratio test fails for the given series.

(b) Similarly for $\sum\left(\frac{(-1)^{n+1}}{n^2}\right)$ we let $a_n=\frac{(-1)^{n+1}}{n^2}$ and the next term $a_{n+1}=\frac{(-1)^{n+2}}{(n+1)^2}$:

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(n+1)^2}\div\frac{(-1)^{n+1}}{n^2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(n+1)^2}\times\frac{n^2}{(-1)^{n+1}}\right|\qquad\left[\begin{array}{l}\text{Turning the second fraction}\\ \text{upside down and multiplying}\end{array}\right]$$

$$=\lim_{n\to\infty}\left|(-1)\left(\frac{n}{n+1}\right)^2\right|\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2=\left(\frac{1}{1+0}\right)^2=1$$

We have $L=1$ therefore the ratio test fails.

(c) Similarly for the given series $\sum\left(\frac{(-1)^{n+1}}{n}\right)$ we can show that $L=1$ which means that the ratio test fails for this series.

---

5. In each case we need to find the interval of convergence.

(a) For the series $\sum_{n=0}^{\infty}\left(nx^{n-1}\right)$ we let $a_n=nx^{n-1}$ therefore $a_{n+1}=(n+1)x^{n+1-1}=(n+1)x^n$.

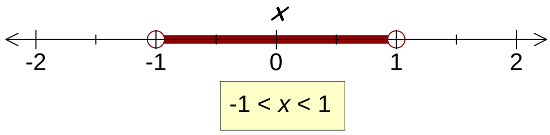
Substituting these into the ratio test $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right|$$

$$=|x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)=|x|\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}_{=1}=|x|$$

Remember the series converges for those values of x which satisfy $L=|x|<1$. This means that for $-1<x<1$ the given series converges. *What happens at the two end points* $x=\pm1$*?*

At $x=\pm1$ we have the series $\sum_{n=0}^{\infty}\left(n(\pm1)^{n-1}\right)$. Since the nth term is $n(\pm1)^{n-1}$ and does **not** converge to zero for large values of n $(n\to\infty)$ so the series diverges at $x=\pm1$.



Hence the interval of convergence is $-1 < x < 1$.

(b) The given series is $\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{\sqrt{n}}\right)$. *What is nth and (n+1) term in this case?*

$a_n = \frac{(-1)^n x^n}{\sqrt{n}}$. Replacing n with $n+1$ into this gives $a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}}$. Substituting these into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ yields

$$
\begin{aligned}
L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}} \div \frac{(-1)^n x^n}{\sqrt{n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}} \times \frac{\sqrt{n}}{(-1)^n x^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)\,x\frac{\sqrt{n}}{\sqrt{n+1}}\right| \qquad \text{[Cancelling like terms]} \\
&= |x|\lim_{n\to\infty}\left(\sqrt{\frac{n}{n+1}}\right) \underbrace{=}_{\substack{\text{Dividing numerator} \\ \text{and denominator by } n}} |x|\underbrace{\lim_{n\to\infty}\left(\sqrt{\frac{1}{1+1/n}}\right)}_{=1} = |x|
\end{aligned}
$$

The series converges for $L = |x| < 1$ which means $-1 < x < 1$. *What happens at* $x = 1$*?*

$$
\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n 1^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^{1/2}}\right)
$$

This series diverges because

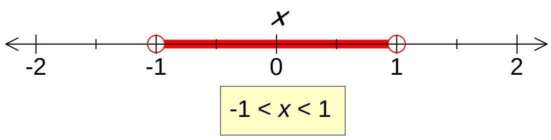
(7.30) (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right)$ converges for $p \ge 1$

In our example we have $p = \frac{1}{2} < 1$.

At $x = -1$ we have

$$
\sum_{n=1}^{\infty}\left(\frac{(-1)^n (-1)^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{2n}}{\sqrt{n}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{1/2}}\right)
$$

Similarly this diverges by the p-test. Our interval of convergence is $-1 < x < 1$.



(c) For the given series $\sum_{n=0}^{\infty}\left(n^p x^n\right)$ we have $a_n = n^p x^n$ and $a_{n+1} = (n+1)^p x^{n+1}$. We need to determine $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^p x^{n+1}}{n^p x^n}\right|$$

$$= |x|\lim_{n\to\infty}\left(\frac{(n+1)^p}{n^p}\right)$$

$$= |x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^p = |x|\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p = |x|(1+0)^p = |x|$$

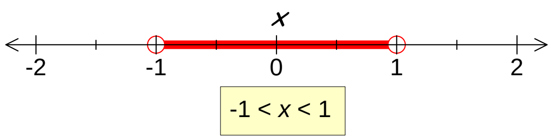
The series converges for $|x|<1$ which means $-1<x<1$. At $x=1$ we have

$$\sum_{n=0}^{\infty}\left(n^p x^n\right) = \sum_{n=0}^{\infty}\left(n^p 1^n\right) = \sum_{n=0}^{\infty}\left(n^p\right)$$

We are given that p is a positive real number so the nth term n^p does **not** tend to zero as $n\to\infty$ therefore the series diverges. At $x=-1$ we have

$$\sum_{n=0}^{\infty}\left(n^p x^n\right) = \sum_{n=0}^{\infty}\left(n^p(-1)^n\right)$$

Similarly because the nth term does **not** tend to zero as $n\to\infty$ therefore the series diverges. Hence the interval of convergence is $-1<x<1$ which we can illustrate as:



(d) Let $a_n = \frac{x^n}{n^2\sqrt{n}} = \frac{x^n}{n^2 n^{1/2}} = \frac{x^n}{n^{5/2}}$ then $a_{n+1} = \frac{x^{n+1}}{(n+1)^{5/2}}$. Substituting these into the ratio test yields:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{5/2}} \div \frac{x^n}{n^{5/2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{5/2}} \times \frac{n^{5/2}}{x^n}\right|$$

$$= |x|\lim_{n\to\infty}\frac{n^{5/2}}{(n+1)^{5/2}}$$

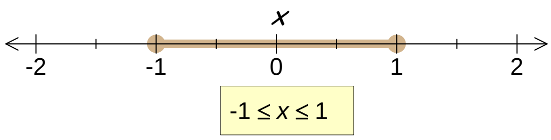
$$= |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{5/2} \underbrace{=}_{\text{Dividing numerator and denominator by } n} \underbrace{|x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{5/2}}_{=1} = |x|$$

The series converges for $|x|<1$ which is equivalent to $-1<x<1$. *What else do we need to consider?*

The two end points $x=\pm 1$. At $x=\pm 1$ we have

$$\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2\sqrt{n}}\right)=\sum_{n=1}^{\infty}\left(\frac{(\pm 1)^n}{n^2 n^{1/2}}\right)=\sum_{n=1}^{\infty}\left(\frac{(\pm 1)^n}{n^{5/2}}\right)$$

This series converges by the p-test because the index of n on the denominator is greater than 1. Hence the interval of convergence is $-1 \le x \le 1$:



(e) This is similar to part (d). Let $a_n=\frac{(-3)^n x^n}{n\sqrt{n}}=\frac{(-3)^n x^n}{nn^{1/2}}=\frac{(-3)^n x^n}{n^{3/2}}$ then

$$a_{n+1}=\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}}$$

Substituting these into the ratio test yields:

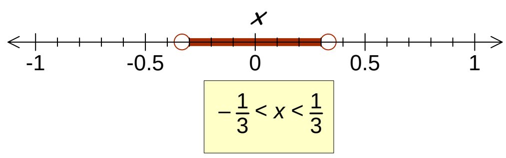
$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}}\div\frac{(-3)^n x^n}{n^{3/2}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}}\times\frac{n^{3/2}}{(-3)^n x^n}\right|$$

$$=\left|(-3)x\right|\lim_{n\to\infty}\frac{n^{3/2}}{(n+1)^{3/2}} \qquad [\text{Cancelling like terms}]$$

$$=3|x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}3|x|\underbrace{\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{3/2}}_{=1}=3|x|$$

The series converges for $3|x|<1$ and dividing by 3 gives $|x|<\frac{1}{3}$. We can illustrate this:



The two end points are $x=\frac{1}{3}$ and $x=-\frac{1}{3}$. At $x=\frac{1}{3}$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-3)^n x^n}{n\sqrt{n}}\right)=\sum_{n=1}^{\infty}\left(\frac{(-3)^n\left(\frac{1}{3}\right)^n}{n(n)^{1/2}}\right)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^{3/2}}\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$$

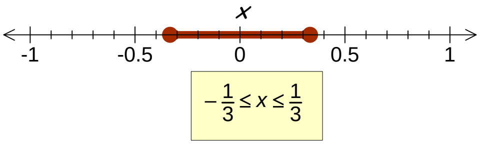
This series converges by the p-test:

(7.30) (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right)$ converges for $p\ge 1$

Similarly at the other end point $x=-\frac{1}{3}$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-3)^n x^n}{n\sqrt{n}}\right)=\sum_{n=1}^{\infty}\left(\frac{(-3)^n\left(-\frac{1}{3}\right)^n}{n(n)^{1/2}}\right)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n(-1)^n}{n^{3/2}}\right)=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

As above this series converges because of the index of n is 3/2 which is greater than 1.



Hence the interval of convergence is $-\frac{1}{3}\le x\le\frac{1}{3}$.

(f) The series under consideration is $\sum_{n=0}^{\infty}\left(x^n n^2 e^{-2n}\right)$. Let $a_n = x^n n^2 e^{-2n}$ then

$$a_{n+1}=x^{n+1}(n+1)^2 e^{-2(n+1)}=x^{n+1}(n+1)^2 e^{-2n-2}$$

The term L is equal to:

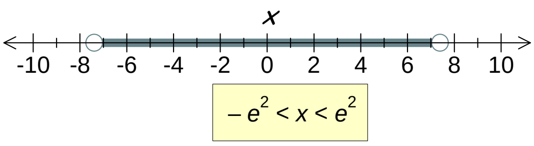
$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)^2 e^{-2n-2}}{x^n n^2 e^{-2n}}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{(n+1)^2 e^{-2n-2}e^{2n}}{n^2}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}e^{-2n-2+2n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2 e^{-2}\right|=\frac{|x|}{e^2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2=\frac{|x|}{e^2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2=\frac{|x|}{e^2}$$

The series converges for $\frac{|x|}{e^2}<1$. Multiplying by e^2 gives $|x|<e^2$:



Our two end points to test are $x=e^2$ and $x=-e^2$. At $x=e^2$ we have

$$\sum_{n=0}^{\infty}\left(x^n n^2 e^{-2n}\right)=\sum_{n=0}^{\infty}\left(\left(e^2\right)^n n^2 e^{-2n}\right)=\sum_{n=0}^{\infty}\left(e^{2n}n^2e^{-2n}\right)\underbrace{=}_{\text{Using the rules of indices}}\sum_{n=0}^{\infty}n^2$$

This series diverges because the nth term does **not** tend to zero. Similarly at $x=-e^2$:

$$\sum_{n=0}^{\infty}\left(x^n n^2 e^{-2n}\right)=\sum_{n=0}^{\infty}\left(\left(-e^2\right)^n n^2 e^{-2n}\right)=\sum_{n=0}^{\infty}\left((-1)^n e^{2n}n^2e^{-2n}\right)\underbrace{=}_{\text{Using the rules of indices}}\sum_{n=0}^{\infty}(-1)^n n^2$$

This also diverges. Hence the interval of convergence is $-e^2<x<e^2$.