Complete solutions to Exercise 1(c)

1. Implication is **only** false if $\neg P$ is true and Q is false for $(\neg P) \Rightarrow Q$.

P	Q	$\neg P$	$(\neg P) \Rightarrow Q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

2. We can construct a truth table to show that a given proposition is a tautology or a contradiction.

(a)

P	$\neg P$	$(\neg P) \vee P$
T	F	T
F	T	T

Hence $(\neg P) \vee P$ is a tautology.

(b)

P	$\neg P$	$(\neg P) \wedge P$
T	F	F
F	T	F

Hence $(\neg P) \wedge P$ is a contradiction.

3. We first make a complete list of all the combinations of P, Q and R and then we evaluate each proposition. Remember $R \wedge (\neg R)$ is a contradiction and therefore is always false.

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$\neg Q$	$P \wedge (\neg Q)$	$R \wedge (\neg R)$	$(P \wedge (\neg Q)) \Rightarrow (R \wedge (\neg R))$	$P \Rightarrow Q$	Col 7 $\Rightarrow$ Col 8
T	T	T	F	F	F	T	T	T
T	T	F	F	F	F	T	T	T
T	F	T	T	T	F	F	F	T
F	T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	F	F	T	T	T

By inspecting the right hand column we can say

$$\left[\left(P \wedge (\neg Q)\right) \Rightarrow \left(R \wedge (\neg R)\right)\right] \Rightarrow (P \Rightarrow Q)$$

is a tautology.

4. (a) We have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	(Col 4) $\wedge$ (Col 5)	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

By looking at the right hand column we can say the following is a tautology.

$$\left[(P \Rightarrow Q) \wedge (P \Rightarrow R)\right] \Rightarrow \left[P \Rightarrow (Q \wedge R)\right]$$

(b) Similarly we have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$R \Rightarrow Q$	(Col 4) $\wedge$ (Col 5)	$P \vee R$	$(P \vee R) \Rightarrow Q$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	F	T	F	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	F	F	T	F	T
F	F	F	T	T	T	F	T	T

Hence

$$\left[(P \Rightarrow Q) \wedge (R \Rightarrow Q)\right] \Rightarrow \left[(P \vee Q) \Rightarrow Q\right] \text{ is a tautology}$$

5.

(a) Let P be 'x is even' and Q be 'x^2 is even'. We have $P \Rightarrow Q$. *What is the contrapositive of* $P \Rightarrow Q$*?*

$$(\neg Q) \Rightarrow (\neg P) \text{ that is 'If } \underbrace{x^2 \text{ is odd}}_{\neg Q} \text{ then } \underbrace{x \text{ is odd}}_{\neg P} \text{ '}$$

The converse is $Q \Rightarrow P$ which is given by 'If $\underbrace{x^2 \text{ is even}}_{Q}$ then $\underbrace{x \text{ is even}}_{P}$'

(b) Similarly: Contrapositive is 'If x is even then x^2 is even' and the converse is 'If x is odd then x^2 is odd'.

6. (i) $P \Rightarrow Q$ means 'If P then Q' which is

'If $x = y$ then $x^2 = y^2$'

This is true.

(ii) The converse of $P \Rightarrow Q$ is the other way round, $Q \Rightarrow P$. This means 'If Q then P'

'If $x^2 = y^2$ then $x = y$'

This is **false** because $x^2 = y^2$ can also give the solutions $x = -y,\ -x = y$ and $-x = -y$.

(iii) The contrapositive of $P \Rightarrow Q$ is $(\neg Q) \Rightarrow (\neg P)$ which is (not Q) implies (not P):

'If $x^2 \neq y^2$ then $x \neq y$'

This is true because implication and its contrapositive have the **same** truth value (they are equivalent).

7. To show the given proposition is a tautology we have to construct a truth table

P	Q	$P \Rightarrow Q$	$\neg Q$	$(P \Rightarrow Q) \wedge (\neg Q)$	$\neg P$	$[(P \Rightarrow Q) \wedge (\neg Q)] \Rightarrow (\neg P)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Since the truth values in the right hand column are **all** true therefore

$$[(P \Rightarrow Q) \wedge (\neg Q)] \Rightarrow (\neg P)$$

is a tautology.

8. (i) The converse is the other way round:

$$(\neg Q) \Rightarrow P$$

(ii) The contrapositive of $P \Rightarrow (\neg Q)$ is $\neg(\neg Q) \Rightarrow (\neg P)$. Remember from section A that

$$\neg(\neg Q) = Q$$

Therefore the contrapositive is equal to $Q \Rightarrow (\neg P)$.

(iii) First we take the contrapositive of $(\neg P) \Rightarrow (\neg Q)$:

$$\neg(\neg Q) \Rightarrow \neg(\neg P)$$

$$Q \Rightarrow P \quad [\text{Remember } \neg(\neg Q) = Q \text{ and } \neg(\neg P) = P]$$

Next we take the converse of this, $Q \Rightarrow P$, that is going the other way, hence $P \Rightarrow Q$.

9. Let P be the proposition '$x > 4$' and Q be the proposition '$x > 0$' then clearly

$P \Rightarrow Q$ which means $x > 4 \quad \Rightarrow \quad x > 0$ is true

But

$Q \Rightarrow P$ which means $x > 0 \quad \Rightarrow \quad x > 4$ is **false** because x could be 3.

The proposition P could have been any number greater than 0.

10. Let P be '$b^2 - 4ac \geq 0$' and Q be 'the equation $ax^2 + bx + c = 0$ has real roots'. We have

$$P \Rightarrow Q$$

Converse is $Q \Rightarrow P$ that is

'If the equation $ax^2 + bx + c = 0$ has real roots then $b^2 - 4ac \geq 0$'

The converse is also true.

Contrapositive is $(\neg Q) \Rightarrow (\neg P)$ that is

'If the equation $ax^2 + bx + c = 0$ does **not** have real roots then $b^2 - 4ac < 0$'

11. Let P be '$1 + a > 0$' and Q be '$(1+a)^n \geq 1 + na$'. We have $P \Rightarrow Q$.

Contrapositive is $(\neg Q) \Rightarrow (\neg P)$ that is

'If $(1+a)^n < 1 + na$ then $1 + a \leq 0$'

Converse is $Q \Rightarrow P$ that is

'If $(1+a)^n \geq 1 + na$ then $1 + a > 0$'