Complete Solutions to Exercise 6(e)

1. (a) We can write the given linear system as $\mathbf{Ax}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix}1&2&1\\3&-3&-2\\4&-2&-5\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}\quad\text{and}\quad \mathbf{b}=\begin{pmatrix}2\\0\\-9\end{pmatrix}$$

and by Cramer's rule we have

$$x=\frac{\det\left(\mathbf{A}_1(\mathbf{b})\right)}{\det(\mathbf{A})},\quad y=\frac{\det\left(\mathbf{A}_2(\mathbf{b})\right)}{\det(\mathbf{A})}\quad\text{and}\quad z=\frac{\det\left(\mathbf{A}_3(\mathbf{b})\right)}{\det(\mathbf{A})}$$

What is $\det(\mathbf{A})$ *equal to?*

$$\begin{aligned}\det(\mathbf{A})&=\det\begin{pmatrix}1&2&1\\3&-3&-2\\4&-2&-5\end{pmatrix}\\&=\det\begin{pmatrix}-3&-2\\-2&-5\end{pmatrix}-2\det\begin{pmatrix}3&-2\\4&-5\end{pmatrix}+\det\begin{pmatrix}3&-3\\4&-2\end{pmatrix}\\&=(15-4)-2(-15+8)+(-6+12)=31\end{aligned}$$

What is $\det\left(\mathbf{A}_1(\mathbf{b})\right)$ *equal to?*

$$\begin{aligned}\det\left(\mathbf{A}_1(\mathbf{b})\right)&=\det\begin{pmatrix}2&2&1\\0&-3&-2\\-9&-2&-5\end{pmatrix}\\&=2\det\begin{pmatrix}-3&-2\\-2&-5\end{pmatrix}-2\det\begin{pmatrix}0&-2\\-9&-5\end{pmatrix}+\det\begin{pmatrix}0&-3\\-9&-2\end{pmatrix}\\&=2(15-4)-2(0-18)+(0-27)=31\end{aligned}$$

What is $\det\left(\mathbf{A}_2(\mathbf{b})\right)$ *equal to?*

$$\begin{aligned}\det\left(\mathbf{A}_2(\mathbf{b})\right)&=\det\begin{pmatrix}1&2&1\\3&0&-2\\4&-9&-5\end{pmatrix}\\&=\det\begin{pmatrix}0&-2\\-9&-5\end{pmatrix}-2\det\begin{pmatrix}3&-2\\4&-5\end{pmatrix}+\det\begin{pmatrix}3&0\\4&-9\end{pmatrix}\\&=(0-18)-2(-15+8)+(-27-0)=-31\end{aligned}$$

What is $\det\left(\mathbf{A}_3(\mathbf{b})\right)$ *equal to?*

$$\begin{aligned}\det\left(\mathbf{A}_3(\mathbf{b})\right)&=\det\begin{pmatrix}1&2&2\\3&-3&0\\4&-2&-9\end{pmatrix}\\&=\det\begin{pmatrix}-3&0\\-2&-9\end{pmatrix}-2\det\begin{pmatrix}3&0\\4&-9\end{pmatrix}+2\det\begin{pmatrix}3&-3\\4&-2\end{pmatrix}\\&=(27-0)-2(-27-0)+2(-6+12)=93\end{aligned}$$

Applying Cramer's rule we have

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{31}{31} = 1$$

$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{-31}{31} = -1$$

$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{93}{31} = 3$$

Our solution to the linear system is $x = 1$, $y = -1$ and $z = 3$.

(b) We can write the given linear system as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 \\ -3 & 5 & -1 \\ 7 & 9 & 13 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ 10 \end{pmatrix}$$

Similar to part (a) we have

$$\det(\mathbf{A}) = \det\begin{pmatrix} 2 & 2 & -1 \\ -3 & 5 & -1 \\ 7 & 9 & 13 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 5 & -1 \\ 9 & 13 \end{pmatrix} - 2\det\begin{pmatrix} -3 & -1 \\ 7 & 13 \end{pmatrix} - \det\begin{pmatrix} -3 & 5 \\ 7 & 9 \end{pmatrix}$$

$$= 2(65+9) - 2(-39+7) - (-27-35) = 274$$

Replacing the first column of **A** by **b** we have the matrix $\mathbf{A}_1(\mathbf{b})$ and the determinant of this matrix is given by

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} 7 & 2 & -1 \\ 0 & 5 & -1 \\ 10 & 9 & 13 \end{pmatrix}$$

$$= 7\det\begin{pmatrix} 5 & -1 \\ 9 & 13 \end{pmatrix} - 0 + 10\det\begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}$$

$$= 7(65+9) + 10(-2+5) = 548$$

Similarly we have

$$\det(\mathbf{A}_2(\mathbf{b})) = \det\begin{pmatrix} 2 & 7 & -1 \\ -3 & 0 & -1 \\ 7 & 10 & 13 \end{pmatrix}$$

$$= 3\det\begin{pmatrix} 7 & -1 \\ 10 & 13 \end{pmatrix} + 0 + \det\begin{pmatrix} 2 & 7 \\ 7 & 10 \end{pmatrix}$$

$$= 3(91+10) + 0 + (20-49) = 274$$

Also

$$\det(\mathbf{A}_3(\mathbf{b})) = \det\begin{pmatrix} 2 & 2 & 7 \\ -3 & 5 & 0 \\ 7 & 9 & 10 \end{pmatrix}$$

$$= 7\det\begin{pmatrix} -3 & 5 \\ 7 & 9 \end{pmatrix} - 0 + 10\det\begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix}$$

$$= 7(-27-35) - 0 + 10(10+6) = -274$$

The solution by Cramer's rule is

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{548}{274} = 2$$

$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{274}{274} = 1$$

$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{-274}{274} = -1$$

Hence we have $x = 2$, $y = 1$ and $z = -1$.

(c) This is very similar to the above parts (a) and (b). We have $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & -6 \\ 1 & -3 & 7 \\ 3 & 4 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

We need to find

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})}, \quad y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} \text{ and } z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}$$

Evaluating each determinant gives:

$$\det(\mathbf{A}) = \det\begin{pmatrix} 1 & 5 & -6 \\ 1 & -3 & 7 \\ 3 & 4 & -2 \end{pmatrix}$$

$$= \det\begin{pmatrix} -3 & 7 \\ 4 & -2 \end{pmatrix} - \det\begin{pmatrix} 5 & -6 \\ 4 & -2 \end{pmatrix} + 3\det\begin{pmatrix} 5 & -6 \\ -3 & 7 \end{pmatrix}$$

$$= (6-28) - (-10+24) + 3(35-18) = 15$$

Next we find $\det(\mathbf{A}_1(\mathbf{b}))$:

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} 1 & 5 & -6 \\ 3 & -3 & 7 \\ 2 & 4 & -2 \end{pmatrix}$$

$$= \det\begin{pmatrix} -3 & 7 \\ 4 & -2 \end{pmatrix} - 3\det\begin{pmatrix} 5 & -6 \\ 4 & -2 \end{pmatrix} + 2\det\begin{pmatrix} 5 & -6 \\ -3 & 7 \end{pmatrix}$$

$$= (6-28) - 3(-10+24) + 2(35-18) = -30$$

We also need $\det(\mathbf{A}_2(\mathbf{b}))$:

$$\det(\mathbf{A}_2(\mathbf{b})) = \det\begin{pmatrix} 1 & 1 & -6 \\ 1 & 3 & 7 \\ 3 & 2 & -2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3 & 7 \\ 2 & -2 \end{pmatrix} - \det\begin{pmatrix} 1 & -6 \\ 2 & -2 \end{pmatrix} + 3\det\begin{pmatrix} 1 & -6 \\ 3 & 7 \end{pmatrix}$$

$$= (-6-14) - (-2+12) + 3(7+18) = 45$$

The last determinant we need to find is $\det(\mathbf{A}_3(\mathbf{b}))$:

$$\det(\mathbf{A}_3(\mathbf{b})) = \det\begin{pmatrix} 1 & 5 & 1 \\ 1 & -3 & 3 \\ 3 & 4 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} -3 & 3 \\ 4 & 2 \end{pmatrix} - \det\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix} + 3\det\begin{pmatrix} 5 & 1 \\ -3 & 3 \end{pmatrix}$$

$$= (-6-12) - (10-4) + 3(15+3) = 30$$

Substituting these values into

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})}, \; y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} \text{ and } z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}$$

gives:

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{-30}{15} = -2$$

$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{45}{15} = 3$$

$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{30}{15} = 2$$

Our solution is $x = -2,\; y = 3$ and $z = 2$.

2. Writing the given system into $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 2 & -1 & -3 & 2 \\ 5 & 3 & 7 & 11 \\ 2 & -4 & 6 & 10 \end{pmatrix}, \; \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 7 \\ 9 \\ 3 \\ 8 \end{pmatrix}$$

What do you notice about the matrix ***A****?*

The bottom row is a multiple of the top row therefore $\det(\mathbf{A}) = 0$. Since $\det(\mathbf{A}) = 0$ we *cannot* use Cramer's rule.

3. (a) We can write the linear system as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 7 \\ 2 & -1 & -3 \\ 7 & 4 & 5 \end{pmatrix}, \; \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$$

The determinant of matrix **A** is

$$\det(\mathbf{A}) = \det\begin{pmatrix} 3 & 2 & 7 \\ 2 & -1 & -3 \\ 7 & 4 & 5 \end{pmatrix}$$
$$= 3\det\begin{pmatrix} -1 & -3 \\ 4 & 5 \end{pmatrix} - 2\det\begin{pmatrix} 2 & -3 \\ 7 & 5 \end{pmatrix} + 7\det\begin{pmatrix} 2 & -1 \\ 7 & 4 \end{pmatrix}$$
$$= 3(-5+12) - 2(10+21) + 7(8+7) = 64$$

What is $\det(\mathbf{A}_1(\mathbf{b}))$ *equal to?*

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} 7 & 2 & 7 \\ -3 & -1 & -3 \\ 5 & 4 & 5 \end{pmatrix}$$

What do you notice about the matrix $\mathbf{A}_1(\mathbf{b})$*?*

The first and last columns of matrix $\mathbf{A}_1(\mathbf{b})$ are identical. *What is the determinant of a matrix which has two identical columns?*

$$\det(\mathbf{A}_1(\mathbf{b})) = 0$$

Similarly the matrix $\mathbf{A}_2(\mathbf{b})$ contains two identical columns therefore

$$\det(\mathbf{A}_2(\mathbf{b})) = 0$$

What about the determinant of $\mathbf{A}_3(\mathbf{b})$*?*

Matrix $\mathbf{A}_3(\mathbf{b})$ is the same as matrix **A** because we replace the third column $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$ with $\mathbf{b} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$. Therefore we have

$$\det(\mathbf{A}_3(\mathbf{b})) = \det(\mathbf{A}) = 64$$

Applying Cramer's rule we have

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{64} = 0$$
$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{64} = 0$$
$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{64}{64} = 1$$

We have $x = 0$, $y = 0$ and $z = 1$.

(b) We can write the linear system as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 11 & -3 & 2 \\ 7 & -1 & 5 \\ 3 & 10 & 8 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}$$

The determinant of matrix **A** is

$$\det(\mathbf{A}) = \det\begin{pmatrix} 11 & -3 & 2 \\ 7 & -1 & 5 \\ 3 & 10 & 8 \end{pmatrix}$$
$$= 11\det\begin{pmatrix} -1 & 5 \\ 10 & 8 \end{pmatrix} + 3\det\begin{pmatrix} 7 & 5 \\ 3 & 8 \end{pmatrix} + 2\det\begin{pmatrix} 7 & -1 \\ 3 & 10 \end{pmatrix}$$
$$= 11(-8-50) + 3(56-15) + 2(70+3) = -369$$

What is $\det(\mathbf{A}_1(\mathbf{b}))$ *equal to?*

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} -3 & -3 & 2 \\ -1 & -1 & 5 \\ 10 & 10 & 8 \end{pmatrix}$$

What do you notice about the matrix $\mathbf{A}_1(\mathbf{b})$*?*

The first two columns of matrix $\mathbf{A}_1(\mathbf{b})$ are identical. *What is the determinant of a matrix which has two identical columns?*

$$\det(\mathbf{A}_1(\mathbf{b})) = 0$$

Similarly the matrix $\mathbf{A}_3(\mathbf{b})$ contains two identical columns therefore $\det(\mathbf{A}_3(\mathbf{b})) = 0$.

What about the determinant of $\mathbf{A}_2(\mathbf{b})$*?*

Matrix $\mathbf{A}_2(\mathbf{b})$ is the same as matrix **A** because the second column, $\begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}$ is replaced by $\mathbf{b} = \begin{pmatrix} -3 \\ -1 \\ 10 \end{pmatrix}$. Therefore we have

$$\det(\mathbf{A}_2(\mathbf{b})) = \det(\mathbf{A}) = -369$$

Applying Cramer's rule we have

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{-369} = 0$$
$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{-369}{-369} = 1$$
$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{-369} = 0$$

The solution of the linear system is $x = 0,\ y = 1$ and $z = 0$.

(c) We can write the linear system as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & 11 \\ -3 & -4 & 6 \\ -2 & 5 & 4 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix}$$

The determinant of matrix **A** is

$$\det(\mathbf{A}) = \det\begin{pmatrix} 5 & 2 & 11 \\ -3 & -4 & 6 \\ -2 & 5 & 4 \end{pmatrix}$$
$$= 5\det\begin{pmatrix} -4 & 6 \\ 5 & 4 \end{pmatrix} - 2\det\begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} + 11\det\begin{pmatrix} -3 & -4 \\ -2 & 5 \end{pmatrix}$$
$$= 5(-16-30) - 2(-12+12) + 11(-15-8) = -483$$

What is $\det(\mathbf{A}_1(\mathbf{b}))$ *equal to?*

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{pmatrix} 5 & 2 & 11 \\ -3 & -4 & 6 \\ -2 & 5 & 4 \end{pmatrix} = \det(\mathbf{A}) = -483$$

Matrix $\mathbf{A}_1(\mathbf{b})$ is the same as matrix **A**.

What about the determinant of $\mathbf{A}_2(\mathbf{b})$*?*

The matrix $\mathbf{A}_2(\mathbf{b})$ contains two identical columns. *What is the determinant of a matrix which has two identical columns?*

$$\det(\mathbf{A}_2(\mathbf{b})) = 0$$

Similarly the matrix $\mathbf{A}_3(\mathbf{b})$ contains two identical columns therefore $\det(\mathbf{A}_3(\mathbf{b})) = 0$.

Applying Cramer's rule we have

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{-483}{-483} = 1$$
$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{-483} = 0$$
$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{0}{-483} = 0$$

The solution of the linear system is $x = 1,\ y = 0$ and $z = 0$.

4. We need to prove that if the jth column of matrix **A** consists of the column **b** and $\det(\mathbf{A}) \neq 0$ then the solution of the system is

$$x_1 = 0,\ x_2 = 0, \cdots,\ x_{j-1} = 0,\ x_j = 1,\ x_{j+1} = 0,\ \cdots \text{ and } x_n = 0$$

Proof. By Cramer's rule we have

$$x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\quad x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}, \ldots\ x_j = \frac{\det(\mathbf{A}_j(\mathbf{b}))}{\det(\mathbf{A})} \ldots \text{ and } x_n = \frac{\det(\mathbf{A}_n(\mathbf{b}))}{\det(\mathbf{A})}$$

We are given that $\det(\mathbf{A}) \neq 0$. *What can we say about* $\det(\mathbf{A}_k(\mathbf{b}))$ *where* $1 \leq k \leq n$ *but* $k \neq j$*?*

Well if $k \neq j$ then the matrix $\mathbf{A}_k(\mathbf{b})$ consists of two identical columns because kth column is the column vector **b** and we are given that the jth column of this matrix is also the column vector **b**. Since $\mathbf{A}_k(\mathbf{b})$ is the matrix with two identical columns therefore the determinant of this matrix is zero, that is we have

$$\det(\mathbf{A}_k(\mathbf{b})) = 0 \text{ provided } k = 1,\ 2,\ 3, \cdots,\ j-1,\ j+1, \cdots,\ n$$

Note this is **only** true if $k \neq j$.

If $k = j$ then the matrix $\mathbf{A}_k(\mathbf{b}) = \mathbf{A}_j(\mathbf{b})$ and this is identical to matrix **A** because the jth column of matrix **A** is the vector column **b**. Therefore we have

$$\det\left(\mathbf{A}_j(\mathbf{b})\right) = \det(\mathbf{A})$$

Substituting these into the above gives

$$x_1 = \frac{\det\left(\mathbf{A}_1(\mathbf{b})\right)}{\det(\mathbf{A})} = \frac{0}{\det(\mathbf{A})} = 0$$

$$x_2 = \frac{\det\left(\mathbf{A}_2(\mathbf{b})\right)}{\det(\mathbf{A})} = \frac{0}{\det(\mathbf{A})} \quad \cdots$$

$$x_j = \frac{\det\left(\mathbf{A}_j(\mathbf{b})\right)}{\det(\mathbf{A})} = \frac{\det(\mathbf{A})}{\det(\mathbf{A})} = 1 \ldots$$

$$x_n = \frac{\det\left(\mathbf{A}_n(\mathbf{b})\right)}{\det(\mathbf{A})} = \frac{0}{\det(\mathbf{A})} = 0$$

This completes our proof.

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5. *Proof*. Since we have $\mathbf{Ax} = \mathbf{O}$ this means that the column vector $\mathbf{b} = \mathbf{O}$. By Cramer's rule we have

$$x_j = \frac{\det\left(\mathbf{A}_j(\mathbf{O})\right)}{\det(\mathbf{A})}$$

where $j = 1, 2, 3, \cdots, n$. *What is* $\det\left(\mathbf{A}_j(\mathbf{O})\right)$ *equal to?*

The matrix $\mathbf{A}_j(\mathbf{O})$ represents a matrix with the all the entries of the jth column are zero.

Hence $\det\left(\mathbf{A}_j(\mathbf{O})\right) = 0$. By Cramer's rule for $j = 1, 2, 3, \cdots, n$ we have

$$x_j = \frac{\det\left(\mathbf{A}_j(\mathbf{O})\right)}{\det(\mathbf{A})} = \frac{0}{\det(\mathbf{A})} = 0$$

This means we have the trivial solution $\mathbf{x} = \mathbf{O}$.

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