Complete Solutions to Exercise 5(e)

1. (a) $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)$. We identify the dominant term and divide numerator and denominator by this dominant term:

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)=\lim_{n\to\infty}\left(\frac{n/n-1/n}{n/n+1/n}\right)\qquad\left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{1-1/n}{1+1/n}\right)\qquad[\text{Cancelling } n\text{'s}]$$

$$=\frac{\lim_{n\to\infty}(1-1/n)}{\lim_{n\to\infty}(1+1/n)}=\frac{1-\lim_{n\to\infty}(1/n)}{1+\lim_{n\to\infty}(1/n)}=\frac{1-0}{1+0}=1$$

(b) $\lim_{n\to\infty}\left(\frac{n-5}{5n+1}\right)$. Evaluating this gives:

$$\lim_{n\to\infty}\left(\frac{n-5}{5n+1}\right)=\lim_{n\to\infty}\left(\frac{n/n-5/n}{5n/n+1/n}\right)\qquad\left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{1-5/n}{5+1/n}\right)\qquad[\text{Cancelling } n\text{'s}]$$

$$=\frac{\lim_{n\to\infty}(1-5/n)}{\lim_{n\to\infty}(5+1/n)}=\frac{1-\lim_{n\to\infty}(5/n)}{5+\lim_{n\to\infty}(1/n)}=\frac{1-0}{5+0}=\frac{1}{5}$$

(c) $\lim_{n\to\infty}\left(\frac{3n+3}{2n-1}\right)$. We have the following:

$$\lim_{n\to\infty}\left(\frac{3n+3}{2n-1}\right)=\lim_{n\to\infty}\left(\frac{3n/n+3/n}{2n/n-1/n}\right)\qquad\left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{3+3/n}{2-1/n}\right)\qquad[\text{Cancelling } n\text{'s}]$$

$$=\frac{\lim_{n\to\infty}(3+3/n)}{\lim_{n\to\infty}(2-1/n)}=\frac{3+\lim_{n\to\infty}(3/n)}{2-\lim_{n\to\infty}(1/n)}=\frac{3+0}{2-0}=\frac{3}{2}$$

(d) $\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{n-5}\right)$. This limit is evaluated by using similar procedures to parts (a), (b) and (c) above:

$$\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{n-5}\right)=\lim_{n\to\infty}\left(\frac{\frac{1}{2}+\frac{3}{n}}{1-\frac{5}{n}}\right)=\frac{\frac{1}{2}+\lim_{n\to\infty}\left(\frac{3}{n}\right)}{1-\lim_{n\to\infty}\left(\frac{5}{n}\right)}$$

$$=\frac{\frac{1}{2}+0}{1-0}=\frac{1}{2}$$

(e) $\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{\frac{1}{5}n-5}\right)$. *What is the dominant term in* $\frac{\frac{1}{2}n+3}{\frac{1}{5}n-5}$ *?*

n. Dividing numerator and denominator by n we have

$$\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{\frac{1}{5}n-5}\right)=\lim_{n\to\infty}\left(\frac{\frac{1}{2}+\frac{3}{n}}{\frac{1}{5}-\frac{5}{n}}\right)$$

$$==\frac{\frac{1}{2}+\lim_{n\to\infty}\left(\frac{3}{n}\right)}{\frac{1}{5}-\lim_{n\to\infty}\left(\frac{5}{n}\right)}=\frac{\frac{1}{2}+0}{\frac{1}{5}+0}=\frac{5}{2}$$

(f) $\lim_{n\to\infty}\left(\frac{\frac{1}{3}n}{\frac{1}{15}n}\right)$. *What do you observe about the term* $\frac{\frac{1}{3}n}{\frac{1}{15}n}$ *?*

n's are common on numerator and denominator so we can cancel out the n's.

$$\lim_{n\to\infty}\left(\frac{\frac{1}{3}\not{n}}{\frac{1}{15}\not{n}}\right)=\lim_{n\to\infty}\left(\frac{\frac{1}{3}}{\frac{1}{15}}\right)=\frac{\frac{1}{3}}{\frac{1}{15}}=\frac{15}{3}=5$$

2. (a) $\lim_{n\to\infty}\left(\frac{2n^2-3}{5n^2+1}\right)$. *What is the dominant term in* $\frac{2n^2-3}{5n^2+1}$ *?*

n^2. Dividing numerator and denominator by n^2 we have

$$\lim_{n\to\infty}\left(\frac{2n^2-3}{5n^2+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{2n^2}{n^2}-\frac{3}{n^2}}{\frac{5n^2}{n^2}+\frac{1}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2-\frac{3}{n^2}}{5+\frac{1}{n^2}}\right) \qquad \left[\text{Cancelling } n^2\right]$$

$$=\frac{2-\lim_{n\to\infty}\left(\frac{3}{n^2}\right)}{5+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}=\frac{2-0}{5+0}=\frac{2}{5}$$

(b) $\lim_{n\to\infty}\left(\frac{7n^2-2n+5}{3n^2+n+4}\right)$. Dividing numerator and denominator by n^2 we have

$$\lim_{n\to\infty}\left(\frac{7n^2-2n+5}{3n^2+n+4}\right)=\lim_{n\to\infty}\left(\frac{\frac{7n^2}{n^2}-\frac{2n}{n^2}+\frac{5}{n^2}}{\frac{3n^2}{n^2}+\frac{n}{n^2}+\frac{4}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{7-\frac{2}{n}+\frac{5}{n^2}}{3+\frac{1}{n}+\frac{4}{n^2}}\right) \qquad \left[\begin{matrix}\text{Cancelling Out}\\ \text{Common Terms}\end{matrix}\right]$$

$$=\frac{7-\lim_{n\to\infty}\left(\frac{2}{n}\right)+\lim_{n\to\infty}\left(\frac{5}{n^2}\right)}{3+\lim_{n\to\infty}\left(\frac{1}{n}\right)+\lim_{n\to\infty}\left(\frac{4}{n^2}\right)}=\frac{7-0+0}{3+0+0}=\frac{7}{3}$$

(c) $\lim_{n\to\infty}\left(\frac{5n^2-n+11}{2n^2+45n-62}\right)$. Similarly we have

$$\lim_{n\to\infty}\left(\frac{5n^2-n+11}{2n^2+45n-62}\right)=\lim_{n\to\infty}\left(\frac{\frac{5n^2}{n^2}-\frac{n}{n^2}+\frac{11}{n^2}}{\frac{2n^2}{n^2}+\frac{45n}{n^2}-\frac{62}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{5-\frac{1}{n}+\frac{11}{n^2}}{2+\frac{45}{n}-\frac{62}{n^2}}\right) \qquad [\text{Cancelling}]$$

$$=\frac{5-\lim_{n\to\infty}\left(\frac{1}{n}\right)+\lim_{n\to\infty}\left(\frac{11}{n^2}\right)}{2+\lim_{n\to\infty}\left(\frac{45}{n}\right)-\lim_{n\to\infty}\left(\frac{62}{n^2}\right)}=\frac{5-0+0}{2+0-0}=\frac{5}{2}$$

(d) $\lim_{n\to\infty}\left(\frac{2n^2-2n+1}{3n^4-2n+1}\right)$. *What is the dominant term in* $\frac{2n^2-2n+1}{3n^4-2n+1}$*?*

n^4. Dividing numerator and denominator by n^4 gives

$$\lim_{n\to\infty}\left(\frac{2n^2-2n+1}{3n^4-2n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{2n^2}{n^4}-\frac{2n}{n^4}+\frac{1}{n^4}}{\frac{3n^4}{n^4}-\frac{2n}{n^4}+\frac{1}{n^4}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{2}{n^2}-\frac{2}{n^3}+\frac{1}{n^4}}{3-\frac{2}{n^3}+\frac{1}{n^4}}\right) \qquad [\text{Cancelling}]$$

$$=\frac{\lim_{n\to\infty}\left(\frac{2}{n^2}\right)-\lim_{n\to\infty}\left(\frac{2}{n^3}\right)+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}{3-\lim_{n\to\infty}\left(\frac{2}{n^3}\right)+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}=\frac{0-0+0}{3-0+0}=0$$

(e) $\lim_{n\to\infty}\left(\dfrac{\frac{1}{3}n^4+1}{7n^4-1}\right)$. The dominant term in $\dfrac{\frac{1}{3}n^4+1}{7n^4-1}$ is n^4 so dividing numerator and denominator by n^4 gives

$$\lim_{n\to\infty}\left(\frac{\frac{1}{3}n^4+1}{7n^4-1}\right)=\lim_{n\to\infty}\left(\frac{\frac{1}{3}\frac{n^4}{n^4}+\frac{1}{n^4}}{\frac{7n^4}{n^4}-\frac{1}{n^4}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{1}{3}+\frac{1}{n^4}}{7-\frac{1}{n^4}}\right)=\frac{\frac{1}{3}+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}{7-\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}=\frac{\frac{1}{3}}{7}=\frac{1}{21}$$

(f) $\lim_{n\to\infty}\left(\dfrac{\frac{9}{2}n^2-n^2+13}{\frac{2}{3}n^2+n-4}\right)$. *Look carefully at the term in the brackets,* $\dfrac{\frac{9}{2}n^2-n^2+13}{\frac{2}{3}n^2+n-4}$.

What do you notice?

We can simplify the numerator because we have like terms and so we can subtract them:

$$\frac{9}{2}n^2-n^2+13=\frac{9}{2}n^2-\frac{2}{2}n^2+13=\frac{7}{2}n^2+13$$

Hence we have

$$\lim_{n\to\infty}\left(\frac{\frac{9}{2}n^2-n^2+13}{\frac{2}{3}n^2+n-4}\right)=\lim_{n\to\infty}\left(\frac{\frac{7}{2}n^2+13}{\frac{2}{3}n^2+n-4}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{7}{2}\frac{n^2}{n^2}+\frac{13}{n^2}}{\frac{2}{3}\frac{n^2}{n^2}+\frac{n}{n^2}-\frac{4}{n^2}}\right)\qquad\left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } n^2\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{\frac{7}{2}+\frac{13}{n^2}}{\frac{2}{3}+\frac{1}{n}-\frac{4}{n^2}}\right)\qquad[\text{Cancelling}]$$

$$=\frac{\frac{7}{2}+\lim_{n\to\infty}\left(\frac{13}{n^2}\right)}{\frac{2}{3}+\lim_{n\to\infty}\left(\frac{1}{n}\right)-\lim_{n\to\infty}\left(\frac{4}{n^2}\right)}=\frac{\frac{7}{2}+0}{\frac{2}{3}+0-0}=\frac{21}{4}$$

3. (a) *How can we evaluate this limit,* $\lim_{n\to\infty}\left(\frac{1}{7^n}\right)$*?*

By rewriting $\frac{1}{7^n}=\left(\frac{1}{7}\right)^n$. Hence by

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

we have

$$\lim_{n\to\infty}\left(\frac{1}{7^n}\right)=\lim_{n\to\infty}\left(\left(\frac{1}{7}\right)^n\right)=0 \qquad \left[\text{Because } x=\frac{1}{7}<1\right]$$

(b) $\lim_{n\to\infty}\left(\left(\frac{1}{6n}\right)^n\right)$. We have $6n>n>1$ therefore $\frac{1}{6n}<1$ and so by

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

with $x=\frac{1}{6n}<1$ we have $\lim_{n\to\infty}\left(\left(\frac{1}{6n}\right)^n\right)=0$.

(c) $\lim_{n\to\infty}\left(\frac{1}{n^n}\right)$.We can rewrite the term in the bracket as $\frac{1}{n^n}=\left(\frac{1}{n}\right)^n$ and since $n>1$ therefore $\frac{1}{n}<1$ and by

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

with $x=\frac{1}{n}<1$ we have $\lim_{n\to\infty}\left(\frac{1}{n^n}\right)=\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n\right)=0$.

(d) $\lim_{n\to\infty}\left(\frac{4^n-1}{5^n+1}\right)$. *What is the dominant term in* $\frac{4^n-1}{5^n+1}$*?*

5^n. Dividing numerator and denominator by 5^n we have

$$\lim_{n\to\infty}\left(\frac{4^n-1}{5^n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{4^n}{5^n}-\frac{1}{5^n}}{\frac{5^n}{5^n}+\frac{1}{5^n}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\left(\frac{4}{5}\right)^n-\frac{1}{5^n}}{1+\frac{1}{5^n}}\right) \qquad [\text{Cancelling}]$$

$$=\frac{\lim_{n\to\infty}\left(\left(\frac{4}{5}\right)^n\right)-\lim_{n\to\infty}\left(\frac{1}{5^n}\right)}{1+\lim_{n\to\infty}\left(\frac{1}{5^n}\right)} \qquad (\dagger)$$

By (5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$ we have these results:

$$\lim_{n\to\infty}\left(\left(\frac{4}{5}\right)^n\right)=0 \text{ and } \lim_{n\to\infty}\left(\frac{1}{5^n}\right)=\lim_{n\to\infty}\left(\left(\frac{1}{5}\right)^n\right)=0$$

Substituting these into (†) gives

$$\lim_{n\to\infty}\left(\frac{4^n-1}{5^n+1}\right)=\frac{0-0}{1+0}=0$$

.

(e) $\lim_{n\to\infty}\left(\dfrac{7^n - n^7 + 7n}{15^n + n + 1}\right)$. *What is the dominant term in* $\dfrac{7^n - n^7 + 7n}{15^n + n + 1}$ *?*

15^n. Dividing numerator and denominator by 15^n and evaluating the limit:

$$\lim_{n\to\infty}\left(\frac{7^n - n^7 + 7n}{15^n + n + 1}\right) = \lim_{n\to\infty}\left(\frac{\frac{7^n}{15^n} - \frac{n^7}{15^n} + \frac{7n}{15^n}}{1 + \frac{n}{15^n} + \frac{1}{15^n}}\right)$$

$$= \frac{\lim_{n\to\infty}\left(\frac{7^n}{15^n}\right) - \lim_{n\to\infty}\left(\frac{n^7}{15^n}\right) + \lim_{n\to\infty}\left(\frac{7n}{15^n}\right)}{1 + \lim_{n\to\infty}\left(\frac{n}{15^n}\right) + \lim_{n\to\infty}\left(\frac{1}{15^n}\right)}$$

$$= \frac{\lim_{n\to\infty}\left(\left(\frac{7}{15}\right)^n\right) - \lim_{n\to\infty}\left(n^7\left(\frac{1}{15}\right)^n\right) + \lim_{n\to\infty}\left(7n\left(\frac{1}{15}\right)^n\right)}{1 + \lim_{n\to\infty}\left(n\left(\frac{1}{15}\right)^n\right) + \lim_{n\to\infty}\left(\left(\frac{1}{15}\right)^n\right)}$$

$$= \frac{0 - 0 + 0}{1 + 0 + 0} = 0 \qquad \left[\text{By (5.19) and (5.20)}\right]$$

(f) $\lim_{n\to\infty}\left(\dfrac{\frac{1}{5^n}}{\frac{1}{3^n}}\right)$. *How can we rewrite* $\dfrac{\frac{1}{5^n}}{\frac{1}{3^n}}$ *?*

$$\frac{\frac{1}{5^n}}{\frac{1}{3^n}} = \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$$

Since $x = \dfrac{3}{5} < 1$ therefore by

(5.19) $\lim_{n\to\infty}\left(x^n\right) = 0$ provided $|x| < 1$

we have

$$\lim_{n\to\infty}\left(\frac{\frac{1}{5^n}}{\frac{1}{3^n}}\right) = \lim_{n\to\infty}\left(\left(\frac{3}{5}\right)^n\right) = 0$$

4. (a) $\lim_{n\to\infty}\left(\dfrac{An^2 + Bn + C}{Dn^2 - En + F}\right)$. Dividing numerator and denominator by n^2 and evaluating the limit gives:

(5.19) $\lim_{n\to\infty}\left(x^n\right) = 0$ provided $|x| < 1$

(5.20) $\lim_{n\to\infty}\left(n^r x^n\right) = 0$ where r is a real number and $|x| < 1$

$$\lim_{n\to\infty}\left(\frac{An^2+Bn+C}{Dn^2-En+F}\right)=\lim_{n\to\infty}\left(\frac{\frac{An^2}{n^2}+\frac{Bn}{n^2}+\frac{C}{n^2}}{\frac{Dn^2}{n^2}-\frac{En}{n^2}+\frac{F}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{A+\frac{B}{n}+\frac{C}{n^2}}{D-\frac{E}{n}+\frac{F}{n^2}}\right) \qquad \text{[Cancelling]}$$

$$=\frac{A+\lim_{n\to\infty}\left(\frac{B}{n}\right)+\lim_{n\to\infty}\left(\frac{C}{n^2}\right)}{D-\lim_{n\to\infty}\left(\frac{E}{n}\right)+\lim_{n\to\infty}\left(\frac{F}{n^2}\right)}=\frac{A+0+0}{D-0+0}=\frac{A}{D}$$

(b) $\lim_{n\to\infty}\left(\frac{Cn^4-2}{Dn^4-2n+1}\right)$. *What is dominant term in* $\frac{Cn^4-2}{Dn^4-2n+1}$ *?*

n^4. Dividing numerator and denominator by n^4 and evaluating the limit we have

$$\lim_{n\to\infty}\left(\frac{Cn^4-2}{Dn^4-2n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{Cn^4}{n^4}-\frac{2}{n^4}}{\frac{Dn^4}{n^4}-\frac{2n}{n^4}+\frac{1}{n^4}}\right)$$

$$=\frac{C-\lim_{n\to\infty}\left(\frac{2}{n^4}\right)}{D-\lim_{n\to\infty}\left(\frac{2}{n^3}\right)+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}=\frac{C-0}{D-0+0}=\frac{C}{D}$$

(c) $\lim_{n\to\infty}\left(\frac{A^n-n^7+7n}{B^n+n+1}\right)$. Since $B>A>1$ therefore the dominant term is B^n.

Dividing numerator and denominator by B^n and evaluating the limit gives:

$$\lim_{n\to\infty}\left(\frac{A^n-n^7+7n}{B^n+n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{A^n}{B^n}-\frac{n^7}{B^n}+7\frac{n}{B^n}}{\frac{B^n}{B^n}+\frac{n}{B^n}+\frac{1}{B^n}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\left(\frac{A}{B}\right)^n-n^7\left(\frac{1}{B}\right)^n+7n\left(\frac{1}{B}\right)^n}{1+n\left(\frac{1}{B}\right)^n+\left(\frac{1}{B}\right)^n}\right)$$

$$=\frac{\lim_{n\to\infty}\left(\left(\frac{A}{B}\right)^n\right)-\lim_{n\to\infty}\left(n^7\left(\frac{1}{B}\right)^n\right)+7\lim_{n\to\infty}\left(n\left(\frac{1}{B}\right)^n\right)}{1+\lim_{n\to\infty}\left(n\left(\frac{1}{B}\right)^n\right)+\lim_{n\to\infty}\left(\left(\frac{1}{B}\right)^n\right)}$$

$$=\frac{0-0+0}{1+0+0}=0$$

5. (a) *How do we find the limit* $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^3\right)$?

We use $\lim_{n\to\infty}(x_n)=L \;\Rightarrow\; \lim_{n\to\infty}\left((x_n)^m\right)=L^m$ with $x_n=\frac{n}{n+1}$:

We first find $\lim_{n\to\infty}(x_n)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$ and then $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^3\right)$. *How?*

What is the dominant term in $x_n=\frac{n}{n+1}$?

n. Dividing numerator and denominator by n and evaluating the limit gives:

$$\lim_{n\to\infty}(x_n)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)=\frac{1}{1+\lim_{n\to\infty}\left(\frac{1}{n}\right)}=\frac{1}{1+0}=1$$

Therefore $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^3\right)=1^3=1$.

(b) $\lim_{n\to\infty}\left(\left(\frac{n^2-2n+1}{2n^2+1}\right)^3\right)$. *What do we evaluate first?*

$\lim_{n\to\infty}\left(\frac{n^2-2n+1}{2n^2+1}\right)$ and then we use $\lim_{n\to\infty}(x_n)=L \;\Rightarrow\; \lim_{n\to\infty}\left((x_n)^m\right)=L^m$ to determine $\lim_{n\to\infty}\left(\left(\frac{n^2-2n+1}{2n^2+1}\right)^3\right)$. *How do we evaluate* $\lim_{n\to\infty}\left(\frac{n^2-2n+1}{2n^2+1}\right)$?

Find the dominant term and then divide numerator and denominator by this term.

What is the dominant term in $\frac{n^2-2n+1}{2n^2+1}$?

n^2. Hence dividing numerator and denominator by n^2 and evaluating the limit gives

$$\lim_{n\to\infty}\left(\frac{n^2-2n+1}{n^2+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{n^2}{n^2}-\frac{2n}{n^2}+\frac{1}{n^2}}{2\frac{n^2}{n^2}+\frac{1}{n^2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1-\frac{2}{n}+\frac{1}{n^2}}{2+\frac{1}{n^2}}\right)\qquad [\text{Cancelling}]$$

$$=\frac{1-\lim_{n\to\infty}\left(\frac{2}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}{2+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}=\frac{1-0+0}{2+0}=\frac{1}{2}$$

Therefore $\lim_{n\to\infty}\left(\left(\frac{n^2-2n+1}{2n^2+1}\right)^3\right)=\left(\frac{1}{2}\right)^3=\frac{1}{8}$.

(c) $\lim_{n\to\infty}\left(\sqrt{\frac{n^4-n^2+1}{7n^4+1}}\right)$. *How can we determine this limit?*

By using $\lim_{n\to\infty}(x_n)=L \quad\Rightarrow\quad \lim_{n\to\infty}\left(\sqrt{x_n}\right)=\sqrt{L}$ with $x_n=\frac{n^4-n^2+1}{7n^4+1}$. Dividing numerator and denominator by n^4 and evaluating the limit we have

$$\lim_{n\to\infty}(x_n)=\lim_{n\to\infty}\left(\frac{n^4-n^2+1}{7n^4+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{n^4}{n^4}-\frac{n^2}{n^4}+\frac{1}{n^4}}{7\frac{n^4}{n^4}+\frac{1}{n^4}}\right)$$

$$=\frac{1-\lim_{n\to\infty}\left(\frac{1}{n^2}\right)+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}{7+\lim_{n\to\infty}\left(\frac{1}{n^4}\right)}=\frac{1}{7}$$

Hence $\lim_{n\to\infty}\left(\sqrt{\frac{n^4-n^2+1}{7n^4+1}}\right)=\sqrt{\frac{1}{7}}=\frac{1}{\sqrt{7}}$.

(d) $\lim_{n\to\infty}\left(\frac{n-1}{\sqrt{3n^2+1}}\right)$. Note that the numerator can be written as $n-1=\sqrt{(n-1)^2}$.

We have $\frac{n-1}{\sqrt{3n^2+1}}=\sqrt{\frac{(n-1)^2}{3n^2+1}}=\sqrt{\frac{n^2-2n+1}{3n^2+1}}$ which means that

$$\lim_{n\to\infty}\left(\frac{n-1}{\sqrt{3n^2+1}}\right)=\lim_{n\to\infty}\left(\sqrt{\frac{n^2-2n+1}{3n^2+1}}\right) \qquad (*)$$

How can we find $\lim_{n\to\infty}\left(\sqrt{\frac{n^2-2n+1}{3n^2+1}}\right)$*?*

By using $\lim_{n\to\infty}(x_n)=L \quad\Rightarrow\quad \lim_{n\to\infty}\left(\sqrt{x_n}\right)=\sqrt{L}$ with $x_n=\frac{n^2-2n+1}{3n^2+1}$. Dividing numerator and denominator by n^2 and working out the limit we have:

$$\lim_{n\to\infty}\left(\frac{n^2-2n+1}{3n^2+1}\right)=\lim_{n\to\infty}\left(\frac{1-\frac{2}{n}+\frac{1}{n^2}}{3+\frac{1}{n^2}}\right)=\frac{1}{3}$$

Hence $\lim_{n\to\infty}\left(\sqrt{\frac{n^2-2n+1}{3n^2+1}}\right)=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$. Substituting this into (*) gives

$$\lim_{n\to\infty}\left(\frac{n-1}{\sqrt{3n^2+1}}\right)=\frac{1}{\sqrt{3}}$$

6. *How do we find* $\lim_{n\to\infty}\left(\cos^n(x)\right)$?

Since $-1<\cos(x)<1$ for $x\neq k\pi$ which means $\left|\cos(x)\right|<1$. We can use

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

Hence by (5.19) we have $\lim_{n\to\infty}\left(\cos^n(x)\right)=\lim_{n\to\infty}\left(\left(\cos(x)\right)^n\right)=0$.

7. This $\lim_{n\to\infty}\left(\dfrac{\sin(n)}{n}\right)$ is very similar to Example 23.

Since $-1\le\sin(n)\le 1$ therefore

$$\frac{-1}{n}\le\frac{\sin(n)}{n}\le\frac{1}{n}$$

and also $\lim_{n\to\infty}\left(\dfrac{-1}{n}\right)=\lim_{n\to\infty}\left(\dfrac{1}{n}\right)=0$. Therefore by the **sandwich rule** we have

$$\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)=0$$

8. (i) *Proof.* Using $\lim_{n\to\infty}(x_n)=0$ and $\lim_{n\to\infty}(y_n)=0$ we have

$$\begin{aligned}\lim_{n\to\infty}(x_n+y_n)&=\lim_{n\to\infty}(x_n)+\lim_{n\to\infty}(y_n)\\&=0+0=0\end{aligned}$$

(ii) *Proof.* Using $\lim_{n\to\infty}(x_n)=0$ and $\lim_{n\to\infty}(y_n)=0$ we have

$$\begin{aligned}\lim_{n\to\infty}(x_ny_n)&=\lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)\\&=0\times 0=0\end{aligned}$$

(iii) *Proof.* Using $\lim_{n\to\infty}(x_n)=0$ and $\lim_{n\to\infty}(y_n)=0$ we have

$$\begin{aligned}\lim_{n\to\infty}(Kx_n)&=K\lim_{n\to\infty}(x_n)\\&=K\times 0=0\end{aligned}$$

9. *Proof.* Since we have to prove $\lim_{n\to\infty}(x_n)=0 \quad\Leftrightarrow\quad \lim_{n\to\infty}\left(|x_n|\right)=0$ that means we have to prove the result both ways, $\Rightarrow$ and $\Leftarrow$.

($\Rightarrow$). Assume $\lim_{n\to\infty}(x_n)=0$ and without loss of generality assume $x_n\ge 0$. Since we have the inequality

$$-x_n\le|x_n|\le x_n$$

and $\lim_{n\to\infty}(x_n)=0,\quad \lim_{n\to\infty}(-x_n)=0$ therefore by the **sandwich rule** we have our result $\lim_{n\to\infty}\left(|x_n|\right)=0$.

($\Leftarrow$). Going the other way we assume $\lim_{n\to\infty}\left(|x_n|\right)=0$ and need to deduce $\lim_{n\to\infty}(x_n)=0$. Since

$$-|x_n|\le x_n\le|x_n|$$

and $\lim_{n\to\infty}\left(-\left|x_n\right|\right)=0,\quad \lim_{n\to\infty}\left(\left|x_n\right|\right)=0$ therefore by the **sandwich rule** we have our required result $\lim_{n\to\infty}\left(x_n\right)=0$.

■

10. *Proof*. Let $\varepsilon>0$ be arbitrary. Since we have $\lim_{n\to\infty}\left(x_n\right)=0$ therefore there is a N_0 such that for all $n>N_0$

$$\left|x_n\right|<\varepsilon_1$$

where $\varepsilon_1>0$ is a function of ε to be determined later.

Consider $\left|\left(x_n\right)^p-0\right|$. We have

$$\left|\left(x_n\right)^p-0\right|=\left(x_n\right)^p<\left(\varepsilon_1\right)^p$$

Let $\varepsilon_1=\varepsilon^{\frac{1}{p}}$ then

$$\left|\left(x_n\right)^p-0\right|<\left(\varepsilon_1\right)^p=\left(\varepsilon^{\frac{1}{p}}\right)^p=\varepsilon$$

Hence $\left|\left(x_n\right)^p-0\right|<\varepsilon$ therefore $\lim_{n\to\infty}\left(\left(x_n\right)^p\right)=0$.

■

11. To determine $\lim_{n\to\infty}\left(\sqrt{n-1}-\sqrt{n}\right)$ we use a trick to rewrite this as follows:

$$\sqrt{n-1}-\sqrt{n}=\left(\sqrt{n-1}-\sqrt{n}\right)\underbrace{\left(\frac{\sqrt{n-1}+\sqrt{n}}{\sqrt{n-1}+\sqrt{n}}\right)}_{=1}\qquad\left[\text{Multiplying by } 1\right]$$

$$=\frac{\left(\sqrt{n-1}-\sqrt{n}\right)\left(\sqrt{n-1}+\sqrt{n}\right)}{\left(\sqrt{n-1}+\sqrt{n}\right)}$$

$$=\frac{n-1-n}{\left(\sqrt{n-1}+\sqrt{n}\right)}\qquad\left[\text{Expanding and Simplifying Numerator}\right]$$

$$=\frac{-1}{\left(\sqrt{n-1}+\sqrt{n}\right)}\qquad\left[\text{Simplifying Numerator}\right]$$

Hence we have $\lim_{n\to\infty}\left(\sqrt{n-1}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\frac{-1}{\left(\sqrt{n-1}+\sqrt{n}\right)}\right)$. *How do we find the Right Hand Side limit,* $\lim_{n\to\infty}\left(\frac{-1}{\left(\sqrt{n-1}+\sqrt{n}\right)}\right)$*?*

Divide numerator and denominator by the dominant term which is $\sqrt{n}$. We have

$$\lim_{n\to\infty}\left(\frac{-1}{\left(\sqrt{n-1}+\sqrt{n}\right)}\right)=\lim_{n\to\infty}\left(\frac{\frac{-1}{\sqrt{n}}}{\left(\frac{\sqrt{n-1}}{\sqrt{n}}+\frac{\sqrt{n}}{\sqrt{n}}\right)}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\frac{-1}{\sqrt{n}}}{\left(\sqrt{1-\frac{1}{n}}+1\right)}\right)\qquad\left[\text{Because } \frac{\sqrt{n-1}}{\sqrt{n}}=\sqrt{\frac{n}{n}-\frac{1}{n}}=\sqrt{1-\frac{1}{n}}\right]$$

$$=\frac{\lim_{n\to\infty}\left(\frac{-1}{\sqrt{n}}\right)}{\lim_{n\to\infty}\left(\sqrt{1-\frac{1}{n}}+1\right)}=\frac{0}{1+1}=0$$

Therefore the given limit is $\lim_{n\to\infty}\left(\sqrt{n-1}-\sqrt{n}\right)=0$.

12. This is difficult and long proof.

Proof. Without loss of generality assume $0<x<1$ because by question 9 above we have $\lim_{n\to\infty}(x_n)=0 \;\Leftrightarrow\; \lim_{n\to\infty}\left(|x_n|\right)=0$ which means that if one sequence is a null sequence then so is the other. We can write $x=\frac{1}{1+y}$ where $y>0$.

First consider the case when $r=1$. We have

$$n^r x^n = nx^n = \frac{n}{(1+y)^n} \qquad (*)$$

Using Binomial for

$$(1+y)^n = 1+ny+\frac{n(n-1)}{2}y^2+\ldots\geq\frac{n(n-1)}{2}y^2$$

Using this inequality with (*) we have

$$\frac{n}{(1+y)^n}\leq\frac{2n}{n(n-1)y^2}$$

The limiting value of the right hand side is given by

$$\lim_{n\to\infty}\left(\frac{2n}{n(n-1)y^2}\right)=\lim_{n\to\infty}\left(\frac{\frac{2n}{n^2}}{\frac{n(n-1)y^2}{n^2}}\right)\qquad\left[\text{Dividing Numerator and Denominator by } n^2\right]$$

$$=\lim_{n\to\infty}\left(\frac{\frac{2}{n}}{\left[1-\frac{1}{n}\right]y^2}\right)=\frac{0}{[1-0]y^2}=0$$

Since for all $n\in\mathbb{N}$, $nx^n\geq 0$ therefore $\lim_{n\to\infty}\left(nx^n\right)\geq 0$ and from above we have $\lim_{n\to\infty}\left(nx^n\right)\leq 0$. Therefore by the **sandwich rule** we have $\lim_{n\to\infty}\left(nx^n\right)=0$.

Consider $r<1$**:** We have $n^r x^n \le nx^n$ and also $0 \le n^r x^n$ which means

$$0 \le n^r x^n \le nx^n$$

Since $\lim_{n\to\infty}(0)=0$ and from above we have $\lim_{n\to\infty}\left(nx^n\right)=0$ therefore by the sandwich rule $\lim_{n\to\infty}\left(n^r x^n\right)=0$ for $r<1$**.**

Consider $r>1$**:** We can rewrite $n^r x^n = \left(n\left(x^{\frac{1}{r}}\right)^n\right)^r$. Since $x^{\frac{1}{r}}<1$ therefore using the first case above we have $\lim_{n\to\infty}\left(n\left(x^{\frac{1}{r}}\right)^n\right)=0$. By Question 10 above we have our required result, $\lim_{n\to\infty}\left(n^r x^n\right)=\lim_{n\to\infty}\left(n\left(x^{\frac{1}{r}}\right)^n\right)^r=0$ for $r>1$.

Hence for any real number r we have

$$\lim_{n\to\infty}\left(n^r x^n\right)=0 \quad \text{provided } |x|<1$$

■