

Complete Solutions to Exercise 1h

1. We need to prove 9 divides $10^n - 1$ for any natural number n .

Proof.

Step 1 The result is true for $n = 1$.

Step 2 Assume the result holds for $n = k$:

$$10^k - 1 = 9m \quad (*)$$

where m is an integer.

Step 3 Required to prove the given proposition holds for $n = k + 1$:

$$10^{k+1} - 1 = 9 \times (\text{integer})$$

Examining the LHS we have

$$\begin{aligned} 10^{k+1} - 1 &= 10(10^k) - 1 \\ &= 9(10^k) + \underbrace{10^k - 1}_{=9m \text{ by } (*)} \quad [\text{Rewriting } 10 = 9 + 1] \\ &= 9(10^k) + 9m = 9(10^k + m) \end{aligned}$$

Hence $10^{k+1} - 1$ is $9 \times (\text{integer})$ so 9 is a divisor of $10^{k+1} - 1$. By mathematical induction we have our result, 9 divides $10^n - 1$ for all $n \in \mathbb{N}$.

2. How do we prove $3 \mid (n^3 - n)$?

Using mathematical induction.

Proof.

The result holds for $n = 1$ because $n^3 - n = 1^3 - 1 = 0$ and $3 \times 0 = 0$ so 3 divides $n^3 - n$ for $n = 1$.

Assume the result is true for $n = k$. This means there is an integer m such that

$$3m = k^3 - k \quad (\dagger)$$

Need to prove $3 \mid ((k+1)^3 - (k+1))$. Expanding the RHS of this gives

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= \underbrace{k^3 - k}_{=3m \text{ by } (\dagger)} + 3k^2 + 3k = 3(m + k^2 + k) \end{aligned}$$

Hence 3 divides $(k+1)^3 - (k+1)$ which means the result holds for $n = k + 1$. By mathematical induction we have our result.

3. *Proof.*

Let $P(n)$ be the given proposition. Check $P(1)$:

$$3 \mid 1(1+1)(1+2) \quad /$$

Assume the result is true for $P(k)$. There exists an integer m such that

$$k(k+1)(k+2) = 3m \quad (*)$$

Using this prove $P(k+1)$, that is required to prove

$$3 \times (\text{integer}) = (k+1)(k+2)(k+3)$$

Expanding the last bracket of the RHS:

$$\underbrace{k(k+1)(k+2)}_{=3m} + 3(k+1)(k+2) = 3m + 3(k+1)(k+2)$$

Hence this is a multiple of 3. Therefore we have proven $P(k+1)$ so by mathematical induction we have our result.

4. To show $n^2 - n$ is an even number we just show by mathematical induction that $2 \mid (n^2 - n)$.

5. *Proof.* We first check the proposition for $n = 1$:

$$a = \frac{a(1-r)}{1-r} = a \quad [\text{Cancelling } (1-r) \text{ 's}]$$

Hence the proposition is true for $n = 1$. *What is our next step?*

Assume the proposition is true for $n = k$, that is

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r} \quad (\$)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$a + ar + ar^2 + \dots + ar^{k-1} + ar^k = \frac{a(1-r^{k+1})}{1-r} \quad (\#)$$

What do we need to prove?

Left Hand Side is equal to the Right Hand side of (#). Examining the Left Hand Side of (#) and using (\$) we have

$$\begin{aligned} a + ar + ar^2 + \dots + ar^{k-1} + ar^k &= \underbrace{a + ar + ar^2 + \dots + ar^{k-1}}_{=\frac{a(1-r^k)}{1-r} \text{ by } (\$)} + ar^k \\ &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k) + ar^k(1-r)}{1-r} && [\text{Common denominator}] \\ &= \frac{a - ar^k + ar^k - ar^k r}{1-r} && [\text{Expanding brackets on numerator}] \\ &= \frac{a - ar^{k+1}}{1-r} && [\text{Because } -ar^k + ar^k = 0] \\ &= \frac{a(1-r^{k+1})}{1-r} && [\text{Factorizing numerator}] \end{aligned}$$

The last line is the Right Hand Side of (#). Therefore we have shown Left Hand Side is equal to the Right Hand side of (#). Hence we have our result.

6. *Proof.*

Clearly the result is true for $n = 1$ because we are given $a \mid b$.

Assume for any natural number k we have

$$a^k \mid b^k \quad (*)$$

Need to prove that $a^{k+1} \mid b^{k+1}$. Examining this we have

$$a^{k+1} = a^k a$$

By (*) $a^k \mid b^k$ and by the initial step $a \mid b$ therefore

$$a^k a \mid b^k b$$

Hence we have our result $a^{k+1} \mid b^{k+1}$. This completes our proof.

7. We cannot assume the result for $n = k + 1$:

$$k + 1 \geq 2^{k+1} - 1 \quad (**)$$

Remember induction says $P(k) \Rightarrow P(k+1)$.

8. (i) We need to prove $3n^2 + 3n + 1 < 2n^3$ for $n \geq 3$.

Proof.

The result is true for $n = 3$ because

$$3(3)^2 + 3(3) + 1 = 37 < 2(3)^3 = 54 \quad /$$

Assume the proposition is true for $n = k$:

$$3k^2 + 3k + 1 < 2k^3 \quad (*)$$

Consider the proposition for the next number $n = k + 1$. Required to prove

$$3(k+1)^2 + 3(k+1) + 1 < 2(k+1)^3 \quad (\dagger)$$

Expanding the LHS of (\dagger) gives

$$\begin{aligned} 3(k+1)^2 + 3(k+1) + 1 &= 3(k^2 + 2k + 1) + 3k + 3 + 1 \\ &= 3k^2 + 6k + 3 + 3k + 3 + 1 \\ &= \underbrace{3k^2 + 3k + 1}_{< 2k^3 \text{ by } (*)} + 6k + 6 < 2k^3 + 6k + 6 \end{aligned}$$

Expanding the RHS of (\dagger) gives

$$\begin{aligned} 2(k+1)^3 &= 2(k^3 + 3k^2 + 3k + 1) \\ &= 2k^3 + 6k^2 + 6k + 2 \end{aligned}$$

Now the LHS \leq RHS $\Leftrightarrow 2k^3 + 6k + 6 < 2k^3 + 6k^2 + 6k + 2 \Leftrightarrow 4 < 6k^2$. Since k is a natural number so $4 < 6k^2$. Hence the inequality in (\dagger) has been established so by mathematical induction we have our given result.

(ii) We need to prove $3^n > n^3$ for $n \geq 4$.

Proof.

Check is true for $n = 4$:

$$3^4 = 81 > 4^3 = 64 \quad /$$

Assume the proposition is true $n = k$, that is

$$3^k > k^3 \quad (*)$$

Required to prove the result for $n = k + 1$, that is

$$3^{k+1} > (k+1)^3 \quad (**)$$

Examining the LHS we have

$$3^{k+1} = 3^k 3 > 3k^3 \quad [\text{By } (*)]$$

Expanding the RHS of (**) gives

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \quad (\dagger)$$

By result of part (i) we have

$$3k^2 + 3k + 1 < 2k^3$$

Substituting this into (\dagger) gives

$$(k+1)^3 = k^3 + \underbrace{3k^2 + 3k + 1}_{< 2k^3} < k^3 + 2k^3 = 3k^3$$

Hence $(k+1)^3 < 3k^3 < 3^k 3 = 3^{k+1}$. This is the inequality in (**) so by mathematical induction we have proven that $3^n > n^3$.

9. (i) You should be able to verify this result.

(ii) Required to prove $2^n \geq n^2$ for $n \geq 4$.

Proof.

Clearly the result is true for $n = 4$ because

$$2^4 = 16 \geq 4^2 = 16$$

Assume the result is true for $n = k$, that is

$$2^k \geq k^2 \quad (*)$$

Required to prove

$$2^{k+1} \geq (k+1)^2 \quad (**)$$

Consider the LHS of this inequality:

$$2^{k+1} = 2^k 2 \geq 2k^2 \quad [\text{By } (*)]$$

We have

$$2k^2 \geq (k+1)^2 \Leftrightarrow 2k^2 \geq k^2 + 2k + 1 \Leftrightarrow k^2 \geq 2k + 1$$

By result of part (i) we have $k^2 \geq 2k + 1$. Hence we have $2^{k+1} \geq 2k^2 \geq (k+1)^2$ which is (**). By mathematical induction we have our required result.

10. We need to prove Bernoulli's inequality;

$$(1+x)^n \geq 1+nx \quad \text{where } x > -1$$

Proof.

Check the result for $n = 1$:

$$(1+x)^1 = 1+x \geq 1+x$$

The result holds for $n = 1$. Assume the result is true for $n = k$:

$$(1+x)^k \geq 1+kx \quad (\dagger)$$

Required to prove

$$(1+x)^{k+1} \geq 1+(k+1)x$$

Using the rules of indices on the LHS:

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k (1+x) \\
&\geq \underbrace{(1+kx)}_{\text{by } (\dagger)} (1+x) \\
&= 1 + (k+1)x + kx^2 \geq 1 + (k+1)x \quad [\text{Because } kx^2 \geq 0]
\end{aligned}$$

Hence the result holds for $n = k + 1$. By mathematical induction we have Bernoulli's inequality.

11. *Proof.* By applying mathematical induction we have:

Check the result is true for $n = 1$, that is

$$\begin{aligned}
\sin(x) &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2(1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \\
&= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \quad (\dagger)
\end{aligned}$$

How do we show the Right Hand Side simplifies to $\sin(x)$?

We need to use the trigonometric identity:

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (\dagger) .

$$\begin{aligned}
\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right) &= -2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right) \\
&= -2\sin(x)\sin\left(-\frac{x}{2}\right) \quad [\text{Simplifying}] \\
&= -2\sin(x)\left(-\sin\left(\frac{x}{2}\right)\right) \quad [\text{Because } \sin(-\theta) = -\sin(\theta)] \\
&= 2\sin(x)\sin\left(\frac{x}{2}\right)
\end{aligned}$$

Substituting this into (\dagger) gives

$$\sin(x) = \frac{2\sin(x)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \sin(x) \quad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right) \right]$$

Hence the proposition is true for $n = 1$. Next we assume the proposition is true for $n = k$:

$$\sin(x) + \sin(2x) + \dots + \sin(kx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \quad (*)$$

We need to prove the proposition for $n = k + 1$, that is

$$\begin{aligned} \sin(x) + \sin(2x) + \dots + \sin(kx) + \sin((k+1)x) &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2(k+1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \\ &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \quad (**) \end{aligned}$$

What do we need to show?

The Left Hand Side is equal to the Right Hand Side of (**). Let's examine the Left Hand Side first.

$$\begin{aligned} \sin(x) + \sin(2x) + \dots + \sin(kx) + \sin((k+1)x) &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} + \sin((k+1)x) \\ &= \frac{\underbrace{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}_{= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by } (**)}}{2\sin\left(\frac{x}{2}\right)} + \sin((k+1)x) \\ &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2k+1}{2}x\right) + 2\sin\left(\frac{x}{2}\right)\sin((k+1)x)}{2\sin\left(\frac{x}{2}\right)} \\ &\quad \text{[Common denominator]} \end{aligned}$$

What do we do next?

We can use the following trigonometric identity on the last term of the numerator:

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

We have

$$\begin{aligned} 2\sin\left(\frac{x}{2}\right)\sin((k+1)x) &= \left[\cos\left(\frac{x}{2} - (k+1)x\right) - \cos\left(\frac{x}{2} + (k+1)x\right)\right] \\ &= \left[\cos\left(\frac{x}{2} - \frac{(2k+2)x}{2}\right) - \cos\left(\frac{x}{2} + \frac{(2k+2)x}{2}\right)\right] \\ &= \left[\cos\left(\frac{x-2kx-2x}{2}\right) - \cos\left(\frac{x+2kx+2x}{2}\right)\right] \\ &= \left[\cos\left(\frac{-x-2kx}{2}\right) - \cos\left(\frac{3x+2kx}{2}\right)\right] \\ &= \left[\cos\left(\frac{x+2kx}{2}\right) - \cos\left(\frac{3x+2kx}{2}\right)\right] \quad \text{[Using } \cos(-\theta) = \cos(\theta)\text{]} \\ &= \left[\cos\left(\frac{(2k+1)x}{2}\right) - \cos\left(\frac{(2k+3)x}{2}\right)\right] \end{aligned}$$

Substituting this into the above we have

$$\begin{aligned} \sin(x) + \sin(2x) + \dots + \sin((k+1)x) &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{(2k+1)x}{2}\right) + \left[\cos\left(\frac{(2k+1)x}{2}\right) - \cos\left(\frac{(2k+3)x}{2}\right) \right]}{2 \sin\left(\frac{x}{2}\right)} \\ &= \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{(2k+3)x}{2}\right)}{2 \sin\left(\frac{x}{2}\right)} \end{aligned}$$

$\left[\text{Because } -\cos\left(\frac{(2k+1)x}{2}\right) + \cos\left(\frac{(2k+1)x}{2}\right) = 0 \right]$. Hence we have the Right Hand

Side of (**). Therefore we have our required result and the proposition is proved by induction.

12. *Proof.* We first check the proposition for $n = 1$

$$(a+b)^1 = a^1 + b^1 = a+b$$

Hence the proposition is true for $n = 1$. *What is our next step?*

Assume the proposition is true for $n = k$, that is

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \dots + b^k \quad (*)$$

We need to prove the proposition for $n = k+1$ which is the following;

$$\begin{aligned} (a+b)^{k+1} &= a^{k+1} + (k+1)a^{k-1+1}b + \frac{(k+1)((k+1)-1)}{2!}a^{(k+1)-2}b^2 + \dots + b^{k+1} \\ &= a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \dots + b^{k+1} \end{aligned}$$

What do we need to show to prove this?

Left Hand Side is equal to the Right Hand Side. *How?*

Using (*) and algebraic manipulation.

