Complete Solutions to Exercise 1g

1. We need to prove the sum of all the even numbers is equal to $n(n+1)$:

$$2+4+6+...+2n=n(n+1)$$

Proof.

We use mathematical induction. Let $P(n)$ be the above proposition. We first check that $P(1)$ is true:

$$2=1(1+1) \qquad \surd$$

We assume the result is true for $n=k$:

$$2+4+6+...+2k=k(k+1) \qquad (*)$$

Required to prove the proposition for $n=k+1$, that is we need to prove

$$2+4+6+...+2k+2(k+1)=(k+1)(k+2)$$

Separating out the LHS of this last equation into the first $2k$ terms and $2(k+1)$ term:

$$\underbrace{2+4+6+...+2k}_{=k(k+1)\text{ by }(*)}+2(k+1)=k(k+1)+2(k+1)$$

$$=(k+1)(k+2) \qquad [\text{Factorizing}]$$

This completes our proof.

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2. Using mathematical induction to prove

$$(1\times 2)+(2\times 3)+(3\times 4)+...+n(n+1)=\frac{1}{3}n(n+1)(n+2)$$

Proof.

Clearly the result is true for $n=1$:

$$(1\times 2)=\frac{1}{3}1(1+1)(1+2)$$

Assume the following is true:

$$(1\times 2)+(2\times 3)+(3\times 4)+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2) \qquad (\dagger)$$

Required to prove that

$$(1\times 2)+(2\times 3)+\cdots+k(k+1)+(k+1)(k+2)=\frac{1}{3}(k+1)(k+2)(k+3)$$

Again splitting the sum on the LHS:

$$\underbrace{(1\times 2)+(2\times 3)+\cdots+k(k+1)}_{=\frac{1}{3}k(k+1)(k+2)\text{ by }(\dagger)}+(k+1)(k+2)=\frac{1}{3}k(k+1)(k+2)+\frac{1}{3}3(k+1)(k+2)$$

$$=\frac{1}{3}(k+1)(k+2)[k+3]$$

This completes our proof.

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3. Very similar to solution 2.

4. We need to prove the proposition $P(n)$ given by

$$1+2+2^2+\cdots+2^n=2^{n+1}-1$$

Proof.
Checking $P(1)$. (Actually we could check $P(0)$.):

$$1+2=2^{1+1}-1 \qquad \surd$$

Assume the result is true for $P(k)$:

$$1+2+2^2+\cdots+2^k=2^{k+1}-1 \qquad (\dagger)$$

Required to prove

$$1+2+2^2+\cdots+2^k+2^{k+1}=2^{k+2}-1 \qquad (\dagger\dagger)$$

Using $(\dagger)$ on the LHS for the sum up to 2^k:

$$\underbrace{1+2+2^2+\cdots+2^k}_{=2^{k+1}-1}+2^{k+1}=2^{k+1}-1+2^{k+1}=2\left(2^{k+1}\right)-1=2^{k+2}-1$$

Hence by mathematical induction we have our result.

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5. This is really the same proposition as question 4. Since we have proven the proposition in question 4 so move the 1 to the right gives us $2+2^2+\cdots+2^n=2^{n+1}-2$.

6. *Proof.* Let $P(n)$ be the given proposition: $2+5+8+...+(3n-1)=\frac{1}{2}n(3n+1)$.

Check $P(1)$. Substituting $n=1$ gives

$$2=\frac{1}{2}(1)(3+1) \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$2+5+8+...+(3k-1)=\frac{1}{2}k(3k+1) \qquad (*)$$

Required to prove the result for $n=k+1$. We need to prove

$$2+5+8+...+(3k-1)+(3(k+1)-1)=\frac{1}{2}(k+1)(3(k+1)+1)$$

$$=\frac{1}{2}(k+1)(3k+4) \qquad (**)$$

How do we prove (**)?
By examining the Left Hand Side and using (*).

$$2+5+\ldots+(3k-1)+(3(k+1)-1)=\underbrace{2+5+8+\ldots+(3k-1)}_{=\frac{1}{2}k(3k+1)\text{ by }(*)}+\underbrace{(3(k+1)-1)}_{=3k+2}$$

$$=\frac{1}{2}k(3k+1)+(3k+2)$$

$$=\frac{1}{2}\left[k(3k+1)+2(3k+2)\right] \qquad \left[\text{Rewriting } (3k+2)=\frac{1}{2}2(3k+2)\right]$$

$$=\frac{1}{2}\left[3k^2+\underbrace{k+6k}_{=7k}+4\right] \qquad [\text{Expanding Brackets}]$$

$$=\frac{1}{2}\left[3k^2+7k+4\right]$$

$$=\frac{1}{2}\left[(k+1)(3k+4)\right] \qquad [\text{Factorizing Quadratic}]$$

The last line is the Right Hand Side of (**). Therefore we have shown (**) and by induction we have our given proposition.

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7. We need to prove that

$$1^3+2^3+3^3+\ldots+n^3=\frac{1}{4}n^2(n+1)^2$$

Proof.

Check $P(1)$:

$$1^3=\frac{1}{4}1^2(1+1)^2 \qquad \surd$$

Assume the result is true for $n=k$:

$$1^3+2^3+3^3+\ldots+k^3=\frac{1}{4}k^2(k+1)^2 \qquad (\dagger)$$

Required to prove

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$$

Separating the sum of the LHS and using (†):

$$\underbrace{1^3+2^3+3^3+\ldots+k^3}_{=\frac{1}{4}k^2(k+1)^2\text{ by }(\dagger)}+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4(k+1)\right] \qquad \left[\text{Factorizing and writing } (k+1)^3=\frac{1}{4}(k+1)^2\,4(k+1)\right]$$

$$=\frac{1}{4}(k+1)^2\left[k^2+4k+4\right]=\frac{1}{4}(k+1)^2[k+2]^2$$

Hence this is our required result.

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8. *Proof.* Let $P(n)$ be the given proposition: $1^3+2^3+3^3+\ldots+n^3=(1+2+3+4+\ldots+n)^2$

Check $P(1)$. Substituting $n=1$ gives

$$1^3=(1)^2 \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = (1 + 2 + 3 + 4 + \ldots + k)^2$$

Required to prove the proposition for $n = k + 1$:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (1 + 2 + 3 + 4 + \ldots + k + (k+1))^2 \qquad (\dagger)$$

Using the given hint on the Left Hand Side of (†) gives

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2 \qquad (\dagger\dagger)$$

[By result of Question 3 with $n = k + 1$]

How do we show this is equal to the Right Hand Side of (†)?

By Example 1 which is

$$1 + 2 + 3 + 4 + \ldots + n = \frac{1}{2}n(n+1)$$

Substituting $n = k + 1$ into this we have

$$1 + 2 + 3 + 4 + \ldots + (k+1) = \frac{1}{2}(k+1)(k+2)$$

Squaring both sides gives

$$(1 + 2 + 3 + 4 + \ldots + (k+1))^2 = \left[\frac{1}{2}(k+1)(k+2)\right]^2$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

This the same as the Right Hand Side of (††). Therefore we have shown (†) which means the result follows by induction.

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9. This is the sum of the finite geometric series:

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} \qquad (r \neq 1)$$

Proof.

Check the result for $n = 1$:

$$1 + r = \frac{1 - r^{1+1}}{1 - r} = \frac{1 - r^2}{1 - r} = \frac{(1-r)(1+r)}{(1-r)} \underbrace{=}_{\text{Cancelling}} 1 + r$$

Hence the result holds for $n = 1$. Assume it is true for $n = k$:

$$1 + r + r^2 + \ldots + r^k = \frac{1 - r^{k+1}}{1 - r} \qquad (*)$$

Required to prove the result for $n = k + 1$:

$$1 + r + r^2 + \cdots + r^k + r^{k+1} = \frac{1 - r^{k+2}}{1 - r} \qquad (**)$$

Separating out the LHS sum:

$$\underbrace{1+r+r^2+\cdots+r^k}_{=\frac{1-r^{k+1}}{1-r} \text{ by (*)}}+r^{k+1}=\frac{1-r^{k+1}}{1-r}+r^{k+1}$$

$$=\frac{1-r^{k+1}+r^{k+1}(1-r)}{1-r}=\frac{1-r^{k+2}}{1-r}$$

Hence we have the RHS of (**) so this completes our proof. By mathematical induction we have the given result.

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10. The first error is in the first line because $P(1)$ is not true.

11. The error is in the last line:

$$p+2k= \text{ prime}$$

For example take $k=\frac{1}{2}p$ then $p+2k=p+p=2p$ which is an even number greater than 2 so cannot be prime.

12. Let $P(n)$ be the given proposition:

$$1^3+3^3+5^3+\cdots+(2n-1)^3=n^2(2n^2-1)$$

Check $P(1)$:

$$1^3=1^2\left(2(1)^2-1\right)$$

Hence the given proposition holds for $n=1$. Assume the following

$$1^3+3^3+5^3+\cdots+(2k-1)^3=k^2(2k^2-1) \qquad (*)$$

Required to prove

$$1^3+3^3+\cdots+(2k-1)^3+(2k+1)^3=(k+1)^2\left(2(k+1)^2-1\right)=(k+1)^2(2k^2+4k+1)$$

Expanding out the RHS gives

$$(k+1)^2(2k^2+4k+1)=2k^4+8k^3+11k^2+6k+1 \qquad (**)$$

Separating out the sum on the LHS and using (*):

$$\underbrace{1^3+3^3+\cdots+(2k-1)^3}_{=k^2(2k^2-1)}+(2k+1)^3=k^2(2k^2-1)+(2k+1)^3$$

Expanding out the RHS of this last equation we have

$$k^2(2k^2-1)+(2k+1)^3=2k^4+8k^3+11k^2+6k+1$$

This is identical to the RHS of (**) which means we have our required result.

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13. *Proof*. Let $P(n)$ be the given proposition:

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^4=\frac{1(1+1)(2+1)(3+3-1)}{30}=\frac{1(2)(3)(5)}{30}=\frac{30}{30}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^4+2^4+3^4+\ldots+k^4=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \qquad (*)$$

Required to prove the proposition for $n=k+1$:

$$1^4+2^4+3^4+\ldots+k^4+(k+1)^4=\frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30}$$

$$=\frac{(k+1)(k+2)(2k+3)(3(k^2+2k+1)+3k+3-1)}{30} \qquad \left[\begin{array}{l}\text{Simplifying}\\ \text{and Expanding}\end{array}\right]$$

$$=\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \qquad (**)$$

Expanding the Left Hand Side of (**) using (*) gives

$$1^4+2^4+3^4+\ldots+k^4+(k+1)^4=\underbrace{1^4+2^4+3^4+\ldots+k^4}_{=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}\text{ by }(*)}+(k+1)^4$$

$$=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4$$

$$=\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]$$

Expanding the square brackets gives:

$$\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)$$

$$=6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30$$

$$=6k^4+39k^3+91k^2+89k+30$$

Left Hand Side of (**) is equal to

$$\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Expanding the Right Hand Side of (**) also gives this result:

$$\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}=\frac{(k+1)}{30}\underbrace{\left[(k+2)(2k+3)(3k^2+9k+5)\right]}_{=6k^4+39k^3+91k^2+89k+30}$$

$$=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Hence the Left Hand Side is equal to the Right Hand Side of (**). We have shown $P(k) \quad \Rightarrow \quad P(k+1)$ therefore our given result follows by induction,

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

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14. *Proof*. Let $P(n)$ be the given proposition:

$$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^5=\frac{1^2(1+1)^2\left(2(1)^2+2(1)-1\right)}{12}=\frac{2^2(2+2-1)}{12}=\frac{4(3)}{12}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^5+2^5+3^5+\ldots+k^5=\frac{k^2(k+1)^2(2k^2+2k-1)}{12} \qquad (€)$$

Required to prove the proposition for $n=k+1$:

$$\begin{aligned}1^5+2^5+3^5+\ldots+k^5+(k+1)^5&=\frac{(k+1)^2\left((k+1)+1\right)^2\left(2(k+1)^2+2(k+1)-1\right)}{12}\\&=\frac{(k+1)^2(k+2)^2\left(2(k^2+2k+1)+2k+2-1\right)}{12}\\&=\frac{(k+1)^2(k+2)^2(2k^2+4k+2+2k+2-1)}{12}\\&=\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12} \qquad (!)\end{aligned}$$

Expanding the Left Hand Side of (!) using (€) gives

$$\begin{aligned}1^5+2^5+3^5+\ldots+k^5+(k+1)^5&=\underbrace{1^5+2^5+3^5+\ldots+k^5}_{=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}}+(k+1)^5\\&=\frac{k^2(k+1)^2(2k^2+2k-1)}{12}+(k+1)^5\\&=\frac{(k+1)^2}{12}\left[k^2(2k^2+2k-1)+12(k+1)^3\right] \quad \left[\text{Taking Out a Common Factor of } \frac{(k+1)^2}{12}\right]\\&=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)\right] \quad [\text{Expanding Brackets}]\\&=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12k^3+36k^2+36k+12\right]\\&=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right] \quad [\text{Collecting Like Terms}]\end{aligned}$$

Expanding the Right Hand Side of (!) gives:

$$\frac{(k+1)^2(k+2)^2(2k^2+6k+3)}{12}=\frac{(k+1)^2}{12}\left[(k+2)^2(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[(k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+6k^3+3k^2+8k^3+24k^2+12k+8k^2+24k+12\right]$$

$$\left[\text{Expanding } (k^2+4k+4)(2k^2+6k+3)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right]$$

Hence the Left Hand Side is equal to the Right Hand Side of (!). We have shown $P(k) \quad \Rightarrow \quad P(k+1)$ therefore our given result follows by induction,

$$1^5+2^5+3^5+\cdots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

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