Show that $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sin(x) - \cos(x)}\,dx = \frac{\pi}{4}$.

18

<u>Soln</u>: We use the following trig identities:

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}$$

where $t = \tan(x/2)$. Differentiating this gives:

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$
$$= \frac{1}{2}\left[1 + \tan^2\left(\frac{x}{2}\right)\right]$$
$$= \frac{1}{2}\left[1+t^2\right]$$
$$dx = \frac{2\,dt}{1+t^2}$$

Consider the rational integrand:

$$\frac{\sin(x)}{\sin(x) - \cos(x)} = \frac{2t}{2t - (1-t^2)}$$
$$= \frac{2t}{t^2+2t-1}$$

Substituting this and $dx = \frac{2\,dt}{1+t^2}$ into the given integral:

$$\int \frac{\sin(x)}{\sin(x) - \cos(x)}\,dx = \int \frac{2t}{t^2+2t-1}\cdot\frac{2}{1+t^2}\,dt$$
$$= \int \frac{4t}{(t^2+2t-1)(t^2+1)}\,dt$$

Writing the integrand as partial fractions we have

$$\frac{4t}{(t^2+2t-1)(t^2+1)} = \frac{At+B}{t^2+2t-1} + \frac{Ct+D}{t^2+1}$$

$$4t = (At+B)(t^2+1) + (Ct+D)(t^2+2t-1)$$

$$= (A+C)t^3 + (B+2C+D)t^2$$
$$+ (A-C+2D)t + (B-D)$$

Equating coeffs.

t^3: $\quad 0 = A+C \quad \Rightarrow \quad A=-C$

t^2: $\quad 0 = B+2C+D \quad (*)$

t: $\quad 4 = A-C+2D \quad (**)$

const: $\quad 0 = B-D \quad \Rightarrow \quad B=D$

Substituting $B=D$ into $(*)$ gives

$$2B+2C=0$$

Substituting $A=-C$ into $(**)$ yields

$$-2C+2D = -2C+2B = 4$$

Adding the last two equations gives

$$4B=4 \Rightarrow B=1$$

Hence $D=B=1$. Substituting $B=D=1$ into $(*)$

$$2+2C=0 \Rightarrow C=-1$$

Hence $A=-C=1$. We have $A=1$, $B=1$, $C=-1$ and $D=1$.

$$\int \frac{4t}{(t^2+2t-1)(t^2+1)}\,dt = \int \frac{t+1}{t^2+2t-1}\,dt + \int \frac{1-t}{t^2+1}$$

$$= \frac{1}{2}\ln|t^2+2t-1| + \int \frac{dt}{t^2+1} - \int \frac{t}{t^2+1}\,dt$$

$$= \frac{1}{2}\ln|t^2+2t-1| + \tan^{-1}(t)$$
$$- \frac{1}{2}\ln(t^2+1)$$
$$+ C$$

We have

$$\int \frac{4t}{(t^2+2t-1)(t^2+1)}\,dt = \frac{1}{2}\ln\left|\frac{t^2+2t-1}{t^2+1}\right| + \tan^{-1}(t) + C$$

Substituting the limits:

$x=0, \quad t=\tan(x/2)=0$

$x=\pi/2, \quad t=\tan\left(\frac{\pi}{4}\right)=1$

We have

$$\int_0^{\pi/2} \frac{\sin(x)}{\sin(x)-\cos(x)}\,dx = \int_0^1 \frac{4t}{(t^2+2t-1)(t^2+1)}\,dt$$

$$= \left[\frac{1}{2}\ln\left|\frac{t^2+2t-1}{t^2+1}\right| + \tan^{-1}(t)\right]_0^1$$

$$= \frac{1}{2}\ln\left(\frac{2}{2}\right) + \tan^{-1}(1) - 0$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$