Chapter 15: Continued Fractions

SECTION A **Finite Continued Fractions**

By the end of this section you will be able to
- convert a rational number into a continued fraction
- understand an application of continued fractions

Continued fractions are another way of representing numbers. *Why use continued fractions to represent numbers?*
Generally examining the decimal expansion of a number lacks symmetry but looking at the same number in continued fractions brings out certain patterns. For example we will find in the next section that the irrational number ® can be approximated by the rational number:

$$\frac{103993}{33102}=3.14159265301190$$

Whilst $® = 3.14159265358979$ $\left(\text{correct to 14 dp}\right)$.

We will find that the fractions $\frac{22}{7}$ and $\frac{355}{113}$ are also good approximations to ®. Both these and the above are produced by continued fractions.
Continued fractions can be used to give rapid approximation to ® by using simple fractions. Indeed, continued fractions are useful for providing rational approximations to real numbers, including irrational numbers. This allows us to express constants such as ®, e and $\sqrt{2}$ in a rational form. The golden ratio $¤ = 1.618034\cdots$ can be expressed in a compact way $\left[1; 1, 1, 1, \cdots\right]$, this notation will be explained later, but notice how beautiful and simple the result is in continued fraction format. No pattern could be extracted from the decimal expansion of $¤ = 1.618034\cdots$, but one can quickly establish one from looking at the continued fraction form. Due to its ability to provide rapid approximations to real numbers, it is common for mathematical computer software to often have continued fractions of certain numbers built into them.
Hence a use for continued fractions is for computing rational approximations to irrational numbers.

A1 Egyptian Fractions

The Egyptian civilization only used what we call unit fractions, that is fractions with a numerator of 1. For example $\frac{3}{4}$ in Egyptian fractions would be written as

$$\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$$

They also did *not* allow repetition of fractions. For example they could *not* write

$$\frac{3}{10}=\frac{1}{10}+\frac{1}{10}+\frac{1}{10}$$

How can we write $\frac{3}{10}$ *as a sum of* ***unit distinct*** *fractions?*

$$\frac{3}{10}=\frac{1}{4}+\frac{1}{20}$$

Similarly we can convert other fractions into distinct unit fractions:

$$\frac{11}{30}=\frac{1}{5}+\frac{1}{6}$$

$$\frac{71}{105}=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$$

Why use Egyptian fractions?

It makes comparing fractions a lot easier. In the above example we can say

$$\frac{11}{30} \text{ is less than } \frac{71}{105}$$

Another example

$$\frac{3}{10}=\frac{1}{4}+\frac{1}{20}$$

$$\frac{7}{15}=\frac{1}{4}+\frac{1}{5}+\frac{1}{60}$$

Clearly $\frac{7}{15}$ is greater than $\frac{3}{10}$. We don't have to convert them to a common denominator in order to answer the question of which fraction is greater.

A2 Introduction to Continued Fractions

We can write the improper fraction

$$\frac{52}{9} \text{ as } 5+\frac{7}{9}$$

We need the numerator to be 1 and *not* 7. *How can we do this?*

Invert $\frac{7}{9}$ that is $\frac{7}{9}=\left(\frac{9}{7}\right)^{-1}$ or $\frac{1}{9/7}$:

$$5+\frac{7}{9}=5+\frac{1}{\frac{9}{7}}$$

But now $\frac{9}{7}$ is an improper fraction. We can write this as $1+\frac{2}{7}$:

$$\frac{52}{9}=5+\frac{7}{9}$$

$$=5+\frac{1}{\frac{9}{7}}=5+\frac{1}{1+\frac{2}{7}}$$

We continue in this manner with only using unit fractions and avoiding improper fractions:

$$\frac{52}{9}=5+\frac{7}{9}$$

$$=5+\frac{1}{\frac{9}{7}}=5+\frac{1}{1+\frac{2}{7}}=5+\frac{1}{1+\frac{1}{\frac{7}{2}}}=5+\frac{1}{1+\frac{1}{3+\frac{1}{2}}}$$

We can write the rational number $\frac{52}{9}$ as $5+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}$. This *staircase* is an example of a continued fraction. Observe that *all* the numerators have a value of 1.

There is a more compact way of writing this continued fraction, $5+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}$. Since *all* the numerators are 1 we ignore these and write the remaining numbers in order within brackets:

$$\frac{52}{9}=5+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}=[5;\ 1,\ 3,\ 2]$$

We can convert any fraction into a continued fraction. The semicolon ; next to the 5 means that this is the integer part of $\frac{52}{9}$ or the floor function $\left\lfloor\frac{52}{9}\right\rfloor=5$.

Which is bigger $\frac{29}{5}$ *or* $\frac{52}{9}$*?*

Well the continued fraction for $\frac{29}{5}=[5;\ 1,\ 4]$. Hence

$$\frac{29}{5} \text{ is greater than } \frac{52}{9}$$

Example 1

Convert $\frac{69}{11}$ into a continued fraction.

<u>Solution</u>

First note that $\frac{69}{11}$ is an improper fraction. Writing $\frac{69}{11}$ as a mixed fraction:

$$\frac{69}{11}=6+\frac{3}{11}$$

Remember for a continued fraction *all* the numerators must be 1. Inverting the last fraction and evaluating the remaining fractions:

$$\frac{69}{11}=6+\frac{3}{11}=6+\frac{1}{\frac{11}{3}}$$

$$=6+\frac{1}{3+\frac{2}{3}} \qquad \left[\text{Writing } \frac{11}{3}=3+\frac{2}{3}\right]$$

$$=6+\frac{1}{3+\frac{1}{\frac{3}{2}}} \qquad \left[\text{Writing } \frac{2}{3}=\frac{1}{3/2}\right]$$

$$=6+\frac{1}{3+\frac{1}{1+\frac{1}{2}}} \qquad \left[\text{Writing } \frac{3}{2}=1+\frac{1}{2}\right]$$

All the numerators are 1 and there is *no* improper fraction so we are done. In compact form we write this as:

$$\frac{69}{11}=[6;\ 3,\ 1,\ 2]$$

For a continued fraction we need to

1) convert all the improper fractions into mixed fractions
2) ensure that all the numerators of any fractions must be 1

Next we give the formal definition of continued fractions.

Definition (15.1).
A **finite continued fraction** is of the following staircase form:

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{\ddots\, a_{n-1}+\cfrac{1}{a_n}}}}}$$

where $a_1,\ a_2,\ a_3,\ a_4,\ \cdots,\ a_n$ are real numbers called the partial quotients of the continued fraction. All of these are positive except possibly a_1.
If *all* the *a*'s are integers then the continued fraction is called a **simple** fraction.

In this chapter we will only deal with simple continued fractions.

Example 2

Express $\frac{450}{743}$ into a continued fraction.

Solution

Note that $\frac{450}{743}$ is *not* an improper fraction. However we can write this as

$$\frac{450}{743}=0+\frac{450}{743}$$

For a continued fraction we can only have numerators of 1. Inverting the given fraction:

$$\frac{450}{743}=0+\frac{450}{743}=0+\cfrac{1}{\cfrac{743}{450}}$$

$$=0+\cfrac{1}{1+\cfrac{293}{450}} \qquad \left[\begin{array}{l}\text{Writing the improper}\\ \text{fraction as a mixed fraction}\end{array}\right]$$

$$=0+\cfrac{1}{1+\cfrac{1}{\cfrac{450}{293}}} \qquad \left[\text{Writing } \frac{293}{450}=\frac{1}{450/293}\right]$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{157}{293}}} \qquad \left[\begin{array}{l}\text{Writing the improper fraction } \frac{450}{293}\\ \text{as a mixed fraction}\end{array}\right]$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{293}{157}}}}$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{136}{157}}}} \qquad \left[\begin{array}{l}\text{Writing the improper fraction } \frac{293}{157}\\ \text{as a mixed fraction}\end{array}\right]$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{157}{136}}}}}$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{21}{136}}}}}$$

Is this ever going to terminate at any point?

Yes because we have a rational which means it must stop eventually. Let us continue with this:

$$0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\frac{21}{136}}}}}}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\cfrac{1}{\frac{136}{21}}}}}}}$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\cfrac{1}{6+\frac{10}{21}}}}}}}$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\cfrac{1}{6+\frac{10}{21}}}}}}}$$

$$=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{\frac{21}{10}}}}}}}}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{2+\frac{1}{10}}}}}}}}$$

How do we write the given fraction $\frac{450}{743}$ *as a continued fraction in compact form?*

$$[0;\ 1,\ 1,\ 1,\ 1,\ 6,\ 2,\ 10]$$

As you can observe in the above example that some fractions take a long time to convert into a continued fraction.

A3 Euclidean Algorithm and Continued Fractions

How do we know that the given fraction $\frac{450}{743}$ *in Example 2 will stop with a finite number of steps?*

The next proposition proves that for a rational number the continued fraction will be finite.

Proposition (15.2).
Any rational number can be written as a finite simple continued fraction.

This means that if we have a rational number then we are guaranteed that only a finite number of steps are required to convert this into a continued fraction.

How do we prove this result?

By using the Euclidean Algorithm.

Proof.

Let $\frac{a}{b}$ be a rational number where $b > 0$. We apply the Euclidean Algorithm which was described in section B3 of chapter 1. (It is the repeated application of the Division Algorithm.)

$$\begin{array}{ll} a = bq_1 + r_1 & 0 < r_1 < b \\ b = r_1 q_2 + r_2 & 0 < r_2 < r_1 \\ r_1 = r_2 q_3 + r_3 & 0 < r_3 < r_2 \\ \vdots & \vdots \\ r_{n-2} = r_{n-1} q_n + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} = r_n q_{n+1} + 0 & 0 < r_n < r_{n-1} \end{array}$$

Using this by replacing the quotients *q*'s with *a*'s gives:

$$\begin{array}{ll} a = ba_1 + r_1 & 0 < r_1 < b \\ b = r_1 a_2 + r_2 & 0 < r_2 < r_1 \\ r_1 = r_2 a_3 + r_3 & 0 < r_3 < r_2 \\ \vdots & \vdots \\ r_{n-2} = r_{n-1} a_n + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} = r_n a_{n+1} + 0 & 0 < r_n < r_{n-1} \end{array}$$

Each of the remainders *r*'s are positive integers. Using the first equation of these results we have

$$\frac{a}{b} = \frac{ba_1 + r_1}{b} = \frac{\not{b}a_1}{\not{b}} + \frac{r_1}{b} = a_1 + \frac{r_1}{b} = a_1 + \cfrac{1}{\cfrac{b}{r_1}} \quad \left[\text{Writing } \frac{r_1}{b} = \frac{1}{b/r_1}\right]$$

Substituting the second equation $b = r_1 a_2 + r_2$ into $\frac{b}{r_1}$ gives

$$\frac{a}{b} = a_1 + \cfrac{1}{\cfrac{b}{r_1}} = a_1 + \cfrac{1}{\cfrac{r_1 a_2 + r_2}{r_1}} = a_1 + \cfrac{1}{a_2 + \cfrac{r_2}{r_1}}$$

If $r_2 = 1$ then we stop and have the resulting continued fraction. If $r_2 \neq 1$ then we continue:

$$\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{r_2}{r_1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cfrac{r_1}{r_2}}}$$

Replacing the third equation $r_1 = r_2 a_3 + r_3$ for r_1 gives:

$$\frac{a}{b}=a_1+\cfrac{1}{a_2+\cfrac{1}{\cfrac{r_1}{r_2}}}=a_1+\cfrac{1}{a_2+\cfrac{1}{\cfrac{r_2a_3+r_3}{r_2}}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{r_3}{r_2}}}$$

We continuing doing this until we get to the zero remainder which is $\frac{r_{n-1}}{r_n}=a_{n+1}$. Hence this means we have

$$\frac{a}{b}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{r_3}{r_2}}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{}{\ddots\cfrac{1}{a_n+a_{n+1}}}}}$$

This completes the proof that for a rational number $\frac{a}{b}$ we only need a finite number of steps to find the continued fraction.

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This proof shows that we can also use the Euclidean Algorithm to find the continued fraction of a rational number.

Example 3

Express the fraction $\frac{450}{743}$ given in the above Example 2 into a continued fraction by using the Euclidean Algorithm.

Solution

Applying the Euclidean Algorithm of $\frac{a}{b}$:

$$\begin{array}{ll} a=bq_1+r_1 & 0<r_1<b \\ b=r_1q_2+r_2 & 0<r_2<r_1 \\ r_1=r_2q_3+r_3 & 0<r_3<r_2 \\ \vdots & \vdots \\ r_{n-2}=r_{n-1}q_n+r_n & 0<r_n<r_{n-1} \\ r_{n-1}=r_nq_{n+1}+0 & 0<r_n<r_{n-1} \end{array}$$

to the given fraction $\frac{450}{743}$ with $a=450$ and $b=743$ gives:

$$450 = 0(743) + 450$$
$$743 = 1(450) + 293$$
$$450 = 1(293) + 157$$
$$293 = 1(157) + 136$$
$$157 = 1(136) + 21$$
$$136 = 6(21) + 10$$
$$21 = 2(10) + 1$$
$$10 = 10(1) + 0$$

Hence the continued fraction in compact form is given by the quotients which are the numbers in front of the brackets. This means that

$$\frac{450}{743} = [0;\ 1,\ 1,\ 1,\ 1,\ 6,\ 2,\ 10]$$

This is identical to the continued fraction found in Example 2.

In this case the Euclidean Algorithm shows that the numbers 450 and 743 are relatively prime, $\gcd(743,\ 450) = 1$.

A4 Applications of Continued Fractions

One application is to see how we can split a rectangle into only squares. The following example illustrates this.

Example 4

Write a 71 by 9 rectangle in terms of squares only.

Compare the three techniques of creating the squares, continued fraction and the quotients in the Euclidean Algorithm.

<u>Solution</u>

We are given the following rectangle:

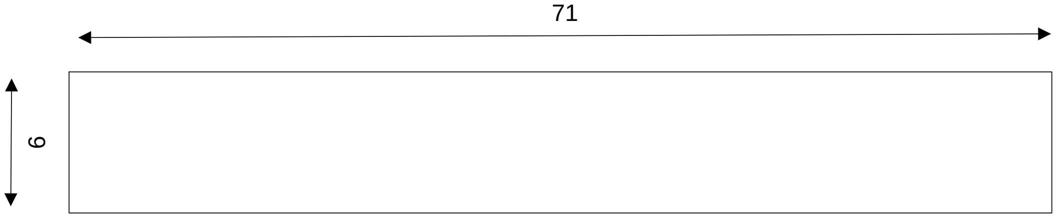

Fig 1

Converting the improper fraction $\frac{71}{9}$ into a mixed fraction we have:

$$\frac{71}{9} = 7 + \frac{8}{9}$$

This means there are 7 squares of size 9 by 9 and one of rectangle of size 8 by 9:

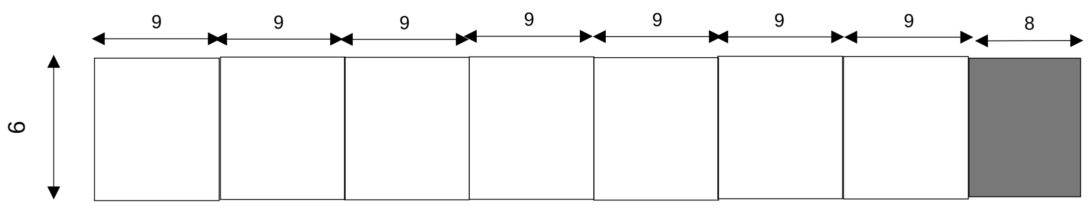

Fig 2

We need to convert the last shaded rectangle into squares.

Evaluating the remaining continued fraction we have

$$\frac{71}{9} = 7 + \frac{8}{9} = 7 + \frac{1}{\frac{9}{8}}$$

$$= 7 + \frac{1}{1 + \frac{1}{8}}$$

This means we have one 8 by 8 square and 8 lots of 1 by 1 squares:

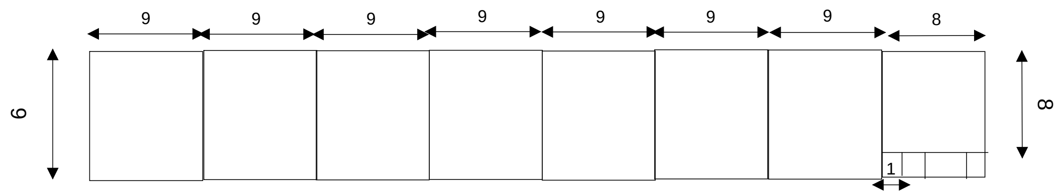

Fig 3

8 lots of one by one squares.

We can write the given fraction $\frac{71}{9}$ as a continued fraction:

$$\frac{71}{9} = [7;\ 1,\ 8]$$

Each of these numbers 7, 1 and 8 correspond to the number of squares in Fig 3. If we apply the Euclidean Algorithm to $\frac{71}{9}$ we obtain the following:

$$71 = 7(9) + 8 \qquad [7 \text{ lots of nine by nine squares}]$$

$$9 = 1(8) + 1 \qquad [1 \text{ eight by eight square}]$$

$$8 = 8(1) + 0 \qquad [8 \text{ lots of one by one squares}]$$

In this algorithm the quotients 7, 1 and 8 correspond to the numbers in the continued fraction and the number of squares in Fig 3.

The following application was lifted from the wonderful article given at the following url; http://plus.maths.org/content/chaos-numberland-secret-life-continued-fractions

Huygens (1629-1695) was interested in building a mechanical model to simulate Saturn's motion with respect to the Earth's motion. At that time it was believed that it took Saturn 29.46 years to orbit the Sun. (Now it is known to take 29.43 years.)

Example 5

Determine the continued fraction of 29.46 and write down the first three rational approximations to this number.

Solution

We have $29.46 = 29\frac{46}{100} = 29 + \frac{46}{100}$. Writing the $\frac{46}{100}$ as a continued fraction:

$$\frac{46}{100} = \frac{1}{100/46} = \frac{1}{2+\frac{8}{46}} = \frac{1}{2+\frac{1}{46/8}} = \frac{1}{2+\frac{1}{5+\frac{6}{8}}} = \frac{1}{2+\frac{1}{5+\frac{1}{8/6}}}$$

$$= \frac{1}{2+\frac{1}{5+\frac{1}{1+\frac{2}{6}}}} = \frac{1}{2+\frac{1}{5+\frac{1}{1+\frac{1}{6/2}}}} = \frac{1}{2+\frac{1}{5+\frac{1}{1+\frac{1}{3}}}}$$

Hence the continued fraction of $29.46 = [29;\ 2,\ 5,\ 1,\ 3]$. The first 3 rational approximations are

$$29.46 \simeq 29$$

$$29.46 \simeq [29;\ 2] = 29 + \frac{1}{2} = \frac{58+1}{2} = \frac{59}{2}$$

$$29.46 \simeq [29;\ 2,\ 5] = 29 + \frac{1}{2+\frac{1}{5}} = 29 + \frac{1}{11/5} = 29 + \frac{5}{11} = \frac{(29\times 11)+5}{11} = \frac{324}{11}$$

The first three approximations are 29, $\frac{59}{2}$ and $\frac{324}{11}$.

SUMMARY

We can convert a rational number into a finite simple continued fraction.

A **finite continued fraction** is of the form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{}{\ddots \; a_{n-1} + \cfrac{1}{a_n}}}}$$

where $a_1,\ a_2,\ a_3,\ a_4,\ \cdots,\ a_n$ are real numbers.