SECTION C **Applying Convergents to Approximate Numbers**

By the end of this section you will be able to

- observe results concerning convergents
- approximate irrational numbers by using convergents

In this section we will demonstrate that convergents of continued fractions provide a good approximation to an irrational number. Remember we are only using the basic arithmetic operations such as addition, subtraction, multiplication and division. No need to use Taylor or Maclaurin series to find approximations to constants such as ®, e, $\sqrt{2}$.
You might like to compare the computation of constants like ® between continued fractions and series solution.
One of the oldest power series used for evaluating ® is

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

This can be used with $x = 1$ because $\tan^{-1}(1) = \frac{®}{4}$.
There are many series which can be used to approximate ®, however they become pretty complex in order to have rapid convergence. With continued fractions we don’t need to use functions such as trigonometric, exponential or logarithmic.
Continued fractions are one of the ways we have of computing values of a constant or of a function to arbitrary precision on a computer.
In the "olden days" before computers, series and continued fractions were used (mostly by groups of women, called "computers") to make numerical tables of functions to a given precision.

C1 Convergents

What does the term convergent mean?
The kth convergent is the evaluation of the continued fraction of the first k terms. It is a rational approximation to the given number.
How did we find the convergents of a given number?
Placing the following results of the last section into a table:

(15.4) $p_n = a_n p_{n-1} + p_{n-2}$

(15.5) $q_n = a_n q_{n-1} + q_{n-2}$

With $p_{-1} = 0$, $q_{-1} = 1$ and $p_0 = 1$, $q_0 = 0$. The nth convergent was given by

$$C_n = \frac{p_n}{q_n}$$

Example 13

(i) Find the first 7 convergents $(C_1, C_2, \cdots, C_7)$ of the irrational number $\sqrt{3} = [1, \langle 1, 2 \rangle]$.

(ii) At each stage find $\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix}$. *What do you notice about your results?*

Solution

(i) We are given the continued fraction $\left[1,\ \langle 1,\ 2\rangle\right]$. *What does this mean?*

$$\left[1,\ \langle 1,\ 2\rangle\right] = [1;\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ \cdots]$$

Remember these entries are the a_n values.

Using the formulae given in the above box and creating a table we have

Step *n*	a_n	p_n	q_n
−1		0	1
0		1	0
1	1	$(1\times1)+0=1$	$(1\times0)+1=1$
2	1	$(1\times1)+1=2$	$(1\times1)+0=1$
3	2	$(2\times2)+1=5$	$(2\times1)+1=3$
4	1	$(1\times5)+2=7$	$(1\times3)+1=4$
5	2	$(2\times7)+5=19$	$(2\times4)+3=11$
6	1	$(1\times19)+7=26$	$(1\times11)+4=15$
7	2	$(2\times26)+19=71$	$(2\times15)+11=41$

Substituting the values of p_n and q_n from the last two columns of the above table we have

$$C_1=\frac{1}{1}=1,\ C_2=\frac{2}{1}=2,\ C_3=\frac{5}{3},\ C_4=\frac{7}{4},\ C_5=\frac{19}{11},\ C_6=\frac{26}{15}\text{ and } C_7=\frac{71}{41}$$

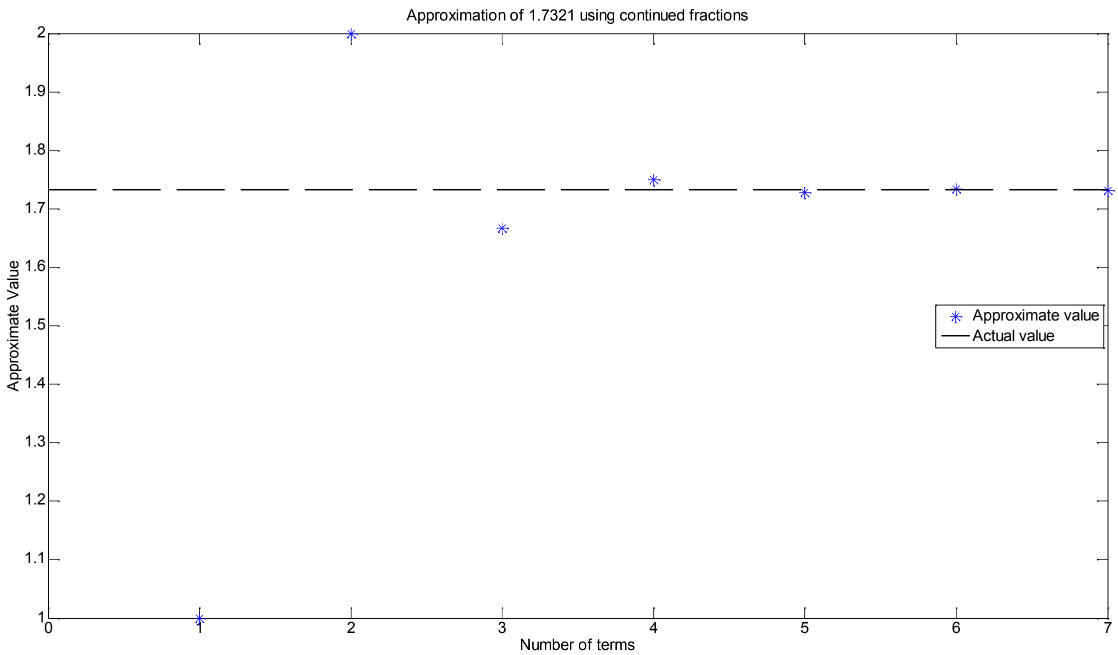

Figure 4

These are the first 7 rational approximations of $\sqrt{3}$ using continued fractions. You may like to check how close each approximation is to $\sqrt{3}$. To find how close the last (C_7) convergent is to $\sqrt{3}$ we use the modulus function. (Note that we only take the modulus so that our error is positive. Since a negative error has no meaning.)

$$\left|\sqrt{3}-\frac{71}{41}\right| = 0.0003435 = \frac{3.435}{10\,000}$$

We conclude that $\frac{71}{41}$ is a reasonable rational approximation for $\sqrt{3}$ as the discrepancy is just over 3 parts in 10 000.

(ii) Substituting $n=1$ into the given formula $\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix}$ yields

$$\det\begin{pmatrix} p_{1-1} & q_{1-1} \\ p_1 & q_1 \end{pmatrix} = \det\begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} \qquad (*)$$

Substituting the values of $p_0=1$, $p_1=1$, $q_0=0$ and $q_1=1$ from the table into (*) gives

$$\det\begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

How do we find the determinant of a matrix?

Well $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$. Using this we have

$$\det\begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (1\times 1)-(0\times 1)=1$$

Working down the table and finding the values of p_n and q_n at each stage we have the following evaluations:

For $n=2$:

We need to find $\det\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = 1-2=-1$

For $n=3$:

$$\det\begin{pmatrix} p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = 6-5=1$$

For $n=4$:

$$\det\begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix} = \det\begin{pmatrix} 5 & 3 \\ 7 & 4 \end{pmatrix} = 20-21=-1$$

For $n=5$:

$$\det\begin{pmatrix} p_4 & q_4 \\ p_5 & q_5 \end{pmatrix} = \det\begin{pmatrix} 7 & 4 \\ 19 & 11 \end{pmatrix} = 77-76=1$$

For $n=6$:

$$\det\begin{pmatrix} p_5 & q_5 \\ p_6 & q_6 \end{pmatrix} = \det\begin{pmatrix} 19 & 11 \\ 26 & 15 \end{pmatrix} = (19\times 15) - (26\times 11) = -1$$

For $n = 7$:

$$\det\begin{pmatrix} p_6 & q_6 \\ p_7 & q_7 \end{pmatrix} = \det\begin{pmatrix} 26 & 15 \\ 71 & 41 \end{pmatrix} = (26\times 41) - (71\times 15) = 1$$

In each case the result is either +1 or -1.

The determinant being equal to ± 1 is *not* a coincidence, nor does it only hold for this example. It is true for continued fractions of *all* numbers.

Proposition (15.8)

Let $\sim = [a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots]$ be a (finite or infinite) continued fraction. If p_n and q_n are defined as above:

(15.4) $p_n = a_n p_{n-1} + p_{n-2}$

(15.5) $q_n = a_n q_{n-1} + q_{n-2}$

Then $\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = p_{n-1}q_n - p_n q_{n-1} = (-1)^{n+1}$.

How do we prove this result?

By using induction.

Proof.

Step 1:

Base case $n = 1$. Let a_1 be our first entry of the continued fraction $[a_1,\ \cdots]$. By our definition we have $p_1 = a_1$ and $q_1 = 1$ because

Step n	a_n	p_n	q_n
-1		0	1
0		1	0
1	a_1	$(a_1\times 1) + 0 = a_1$	$(a_1\times 0) + 1 = 1$

For $n = 1$ we have the determinant of the highlighted values:

$$\det\begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} = 1$$

Step 2:

Assume the result is true for $n = m$, that is

$$\det\begin{pmatrix} p_{m-1} & q_{m-1} \\ p_m & q_m \end{pmatrix} = p_{m-1}q_m - p_m q_{m-1} = (-1)^{m+1} \qquad (\dagger)$$

Step 3:

Required to prove the result for $n = m + 1$:

$$\det\begin{pmatrix} p_m & q_m \\ p_{m+1} & q_{m+1} \end{pmatrix} = (-1)^{m+2}$$

Writing out the previous steps in a table we have:

Step *n*	a_n	p_n	q_n
$\vdots$		$\vdots$	$\vdots$
$m-1$	a_{m-1}	p_{m-1}	q_{m-1}
$\boldsymbol{m}$	a_m	p_m	q_m
$m+1$	a_{m+1}	$a_{m+1}p_m + p_{m-1}$	$a_{m+1}q_m + q_{m-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The determinant we want to examine is highlighted in the table:

$$\det\begin{pmatrix} p_m & q_m \\ p_{m+1} & q_{m+1} \end{pmatrix} = \det\begin{pmatrix} p_m & q_m \\ a_{m+1}p_m + p_{m-1} & a_{m+1}q_m + q_{m-1} \end{pmatrix}$$

Evaluating this determinant:

$$\begin{aligned}\det\begin{pmatrix} p_m & q_m \\ a_{m+1}p_m + p_{m-1} & a_{m+1}q_m + q_{m-1} \end{pmatrix} &= p_m\left(a_{m+1}q_m + q_{m-1}\right) - q_m\left(a_{m+1}p_m + p_{m-1}\right) \\ &= a_{m+1}\underbrace{\left(p_m q_m - q_m p_m\right)}_{=0} + \left[p_m q_{m-1} - q_m p_{m-1}\right] \qquad \text{[Factorising]} \\ &= 0 - \left[q_m p_{m-1} - p_m q_{m-1}\right] \\ &= -\det\begin{pmatrix} p_{m-1} & q_{m-1} \\ p_m & q_m \end{pmatrix} = (-1)\underbrace{(-1)^{m+1}}_{\text{By } (\dagger)} = (-1)^{m+2}\end{aligned}$$

This is our required result. By mathematical induction we have

$$\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = (-1)^{n+1}$$

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We can use this proposition to solve Diophantine equations.

Example 14

(i) Find the simple continued fraction of $\dfrac{111}{53}$.

(ii) Solve the Diophantine equation

$$111x + 53y = 10$$

<u>Solution</u>

(i) Using the Euclidean algorithm to find the continued fraction:

$$\begin{aligned} 111 &= 2(53) + 5 \\ 53 &= 10(5) + 3 \\ 5 &= 1(3) + 2 \\ 3 &= 1(2) + 1 \\ 2 &= 2(1) \end{aligned}$$

Hence the continued fraction is given by the quotients which are the numbers in front of the brackets:

$$\frac{111}{53} = [2;\ 10,\ 1,\ 1,\ 2]$$

(ii) The convergents C_1, C_2, C_3, C_4 and C_5 are given by:

Step n	a_n	p_n	q_n
−1		0	1
0		1	0
1	2	$(2\times 1)+0=2$	$(2\times 0)+1=1$
2	10	$(10\times 2)+1=21$	$(10\times 1)+0=10$
3	1	$(1\times 21)+2=23$	$(1\times 10)+1=11$
4	1	$(1\times 23)+21=44$	$(1\times 11)+10=21$
5	2	$(2\times 44)+23=111$	$(2\times 21)+11=53$

Using the above Proposition (15.8):

$$\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = p_{n-1}q_n - p_n q_{n-1} = (-1)^{n+1}$$

Finding the determinant of the highlighted entries:

$$\det\begin{pmatrix} 44 & 21 \\ 111 & 53 \end{pmatrix} = 53(44) - 111(21) = (-1)^{5+1} = 1$$

We can write this $53(44)-111(21)=1$ as

$$111(-21)+53(44)=1 \qquad (*)$$

However we need to solve $111x+53y=10$. *How do we find x and y?*

Multiply (*) by 10:

$$111(-210)+53(440)=10$$

Hence a particular solution is $x_0=-210$ and $y_0=440$. *How do we find the general solution?*

Using Corollary (1.18) of chapter 1:

Let $\gcd(a,\ b)=1$ and x_0, y_0 be particular solutions of the equation

$$ax+by=c$$

Then *all* the other solutions of this equation are given by

$$x = x_0 + bt \text{ and } y = y_0 - at$$

where t is any integer.

In the above example we used corollary (1.18). To be able to use the above to do that we needed $\gcd(111,\ 53)$ to be 1, which it is. Thankfully, as the next corollary will show, this is always the case for p_n and q_n. Therefore, we can always use corollary (1.18) in these circumstances.

We need to find the general solution of $111x+53y=10$.

Using this corollary with $a=111$, $b=53$, $x_0=-210$ and $y_0=440$ we have

$$x = x_0 + bt = -210 + 53t \text{ and } y = y_0 - at = 440 - 111t$$

Hence our general solution is $x = -210 + 53t$ and $y = 440 - 111t$.

Corollary (15.9).

Let $\sim = [a_1;\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots]$ be a (finite or infinite) continued fraction. If p_n and q_n are defined as above:

(15.4) $p_n = a_n p_{n-1} + p_{n-2}$

(15.5) $q_n = a_n q_{n-1} + q_{n-2}$

Then p_n and q_n are relatively prime.

What does relatively prime mean?

It means that p_n and q_n have *no* factors in common apart from 1 or $\gcd(p_n,\ q_n) = 1$.

Proof.

Let $g = \gcd(p_n,\ q_n)$. We are required to prove that $g = 1$.

This means that $g \mid p_n$ and $g \mid q_n$ and moreover g divides any linear combination of these.

Hence g divides $p_{n-1}q_n - p_n q_{n-1}$. By the above Proposition (15.8):

$$\det\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = p_{n-1}q_n - p_n q_{n-1} = (-1)^{n+1}$$

This implies that $g \mid (-1)^{n+1}$. By the definition of the gcd we have $g > 0$ therefore

$$g = 1$$

This completes the proof that p_n and q_n are relatively prime.

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C2 Rational Approximations to Irrational Numbers

A rational approximation to e is $\frac{19}{7}$ which is given by the continued fraction $[2;\ 1,\ 2,\ 1,\ 1]$.

Graphically this is shown below:

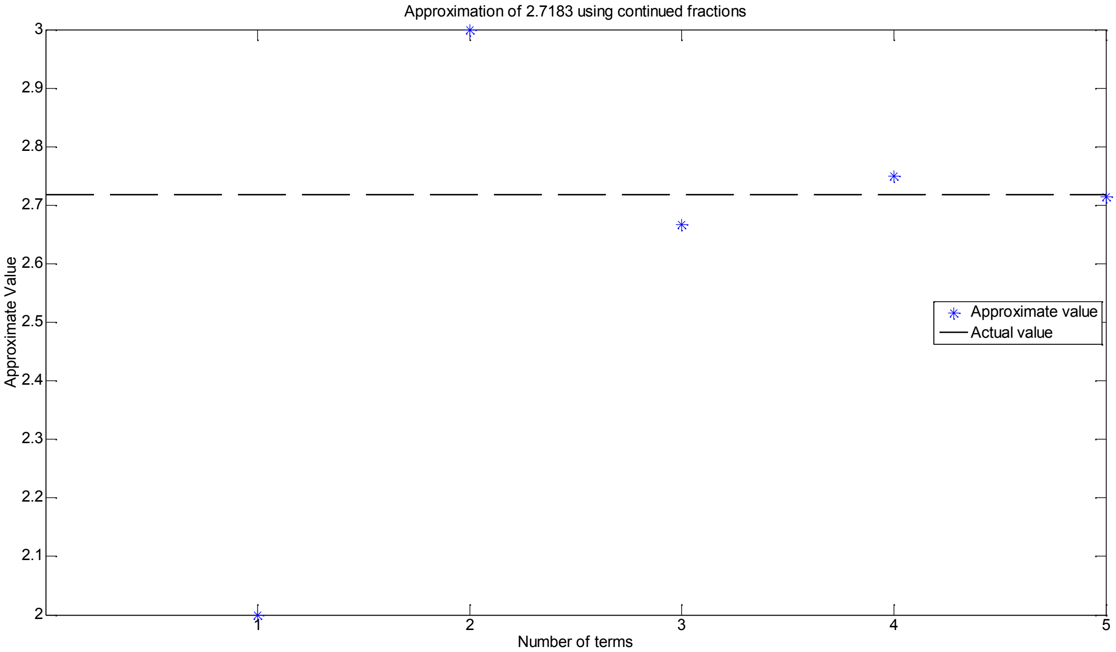

Figure 5

Note that the convergent values get closer as the number of terms is increased. However, it is useful for us to quantify the difference between the value of the kth convergence and the actual value.

How close is this approximation 19/7?

Well we can find the error between e and 19/7 by using the modulus function:

$$\left|e-\frac{19}{7}\right| = 4\times 10^{-3}(1\text{sf}) \text{ or } \frac{1}{250}$$

19/7 is *not* a bad approximation to e as the discrepancy is 1 part in 250.

Lemma (15.10).

Let $\sim = [a_1;\ a_2,\ a_3,\ \cdots,\ a_n,\ \sim_{n+1}]$ be an infinite continued fraction. Let p_n and q_n be defined as above in (15.4) and (15.5).

For all natural numbers n we have

$$\sim - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_n(\sim_{n+1}q_n + q_{n-1})}$$

What does this mean?

The difference between the given irrational number $\sim$ and the nth convergent (nth rational approximation) $\frac{p_n}{q_n}$ is equal to the right hand term; $\frac{(-1)^{n+1}}{q_n(\sim_{n+1}q_n + q_{n-1})}$.

Proof.

Exercise 15(c).

Convergent Approximation Theorem (15.11).

Let $\sim = [a_1;\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots]$ be an infinite continued fraction. Let p_n and q_n be defined as above in (15.4) and (15.5).

For all natural numbers *n* we have

$$\left|\sim - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

What does this mean?

The rational approximation to $\sim$ is less than $\frac{1}{q_n q_{n+1}}$. This gives us an idea of how accurate our approximation is.

Proof.

By the above Lemma (15.10) and using $a_{n+1} = \lfloor \sim_{n+1} \rfloor < \sim_{n+1}$ because we have an infinite continued fraction $\sim$. Therefore

$$\left|\sim - \frac{p_n}{q_n}\right| = \left|\frac{(-1)^{n+1}}{q_n(\sim_{n+1}q_n + q_{n-1})}\right|$$

$$= \frac{1}{q_n(\sim_{n+1}q_n + q_{n-1})} \underset{\text{Because } a_{n+1} < \sim_{n+1}}{\leq} \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \quad [\text{Because } \ a_{n+1}q_n + q_{n-1} = q_{n+1}]$$

This completes our proof.

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Example 15

The first five terms of the continued fraction for ® are $[3;\ 7,\ 15,\ 1,\ 292]$.

Estimate the accuracy of C_1, C_2, C_3 and C_4 as an approximation to ® by giving an upper bound for $|® - C_i|$ where $i = 1,\ 2,\ 3,\ 4$.

<u>Solution</u>

Creating the table:

Step *n*	a_n	p_n	q_n
−1		0	1
0		1	0
1	3	$(3\times1)+0=3$	$(3\times0)+1=1$
2	7	$(7\times3)+1=22$	$(7\times1)+0=7$
3	15	$(15\times22)+3=333$	$(15\times7)+1=106$
4	1	$(1\times333)+22=355$	$(1\times106)+7=113$
5	292	$(292\times355)+333=103993$	$(292\times113)+106=33102$

Applying the above result (15.11):

$$\left| \sim - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

Remember the q values are in the last column and $C_n = \frac{p_n}{q_n}$. The upper bound at various convergents are:

$$\left| \circledR - \frac{p_1}{q_1} \right| = \left| \circledR - \frac{3}{1} \right| < \frac{1}{1 \times 7} = \frac{1}{7}$$

$$\left| \circledR - \frac{p_2}{q_2} \right| = \left| \circledR - \frac{22}{7} \right| < \frac{1}{7 \times 106} = \frac{1}{742}$$

$$\left| \circledR - \frac{p_3}{q_3} \right| = \left| \circledR - \frac{333}{106} \right| < \frac{1}{106 \times 113} = \frac{1}{11\,978}$$

$$\left| \circledR - \frac{p_4}{q_4} \right| = \left| \circledR - \frac{355}{113} \right| < \frac{1}{113 \times 33\,102} = \frac{1}{3\,740\,526}$$

One can easily see that the error improves rapidly by looking at the figure below. The red lines are error bars between the “actual” value and the value at the given convergent. Notice how small the second error bar is compared to the first and how quickly they shrink to sizes where comparison becomes difficult to do easily.

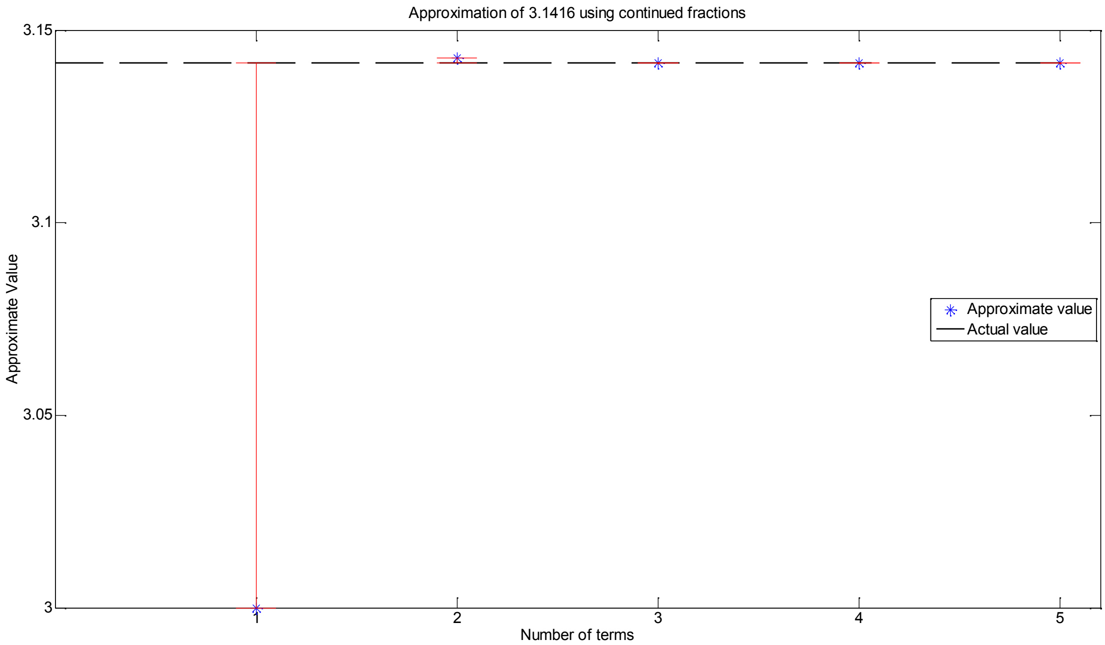

Figure 6

$\frac{355}{113}$ is a Chinese approximation to ® because it was discovered by the ancient Chinese mathematician Zu Chongzhi(429 AD-500AD). This approximation is within about 1 in 4 million.

In the 15^{th} century, Arabic mathematician Al-Kashi managed to get got an approximation to ® with an accuracy of 16 decimal places.

SUMMARY
We approximate irrational numbers by using continued fractions. Our rational approximation is less than $\frac{1}{q_n q_{n+1}}$.