MT1

Admin

Admin

- It is **vital** that you arrive at the exam room on time. If you are not in your seat you risk missing the exam and an automatic fail.

MT1 Semester B Lecture 11

Improper integration

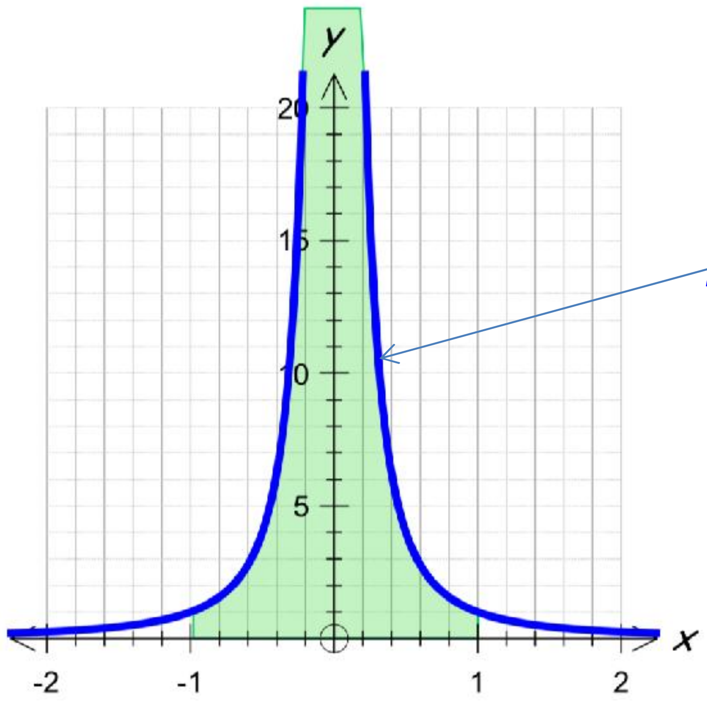
Improper integration

- An integral is said to be improper if
  - The integrand has a singularity (goes to infinity) at some point in the range over which we're integrating, or
  - The range of the integral is infinite.

Improper Integrals

- Determine the shaded area and the integral:

$$\int_{-1}^{1} x^{-2}\,\mathrm{d}x$$



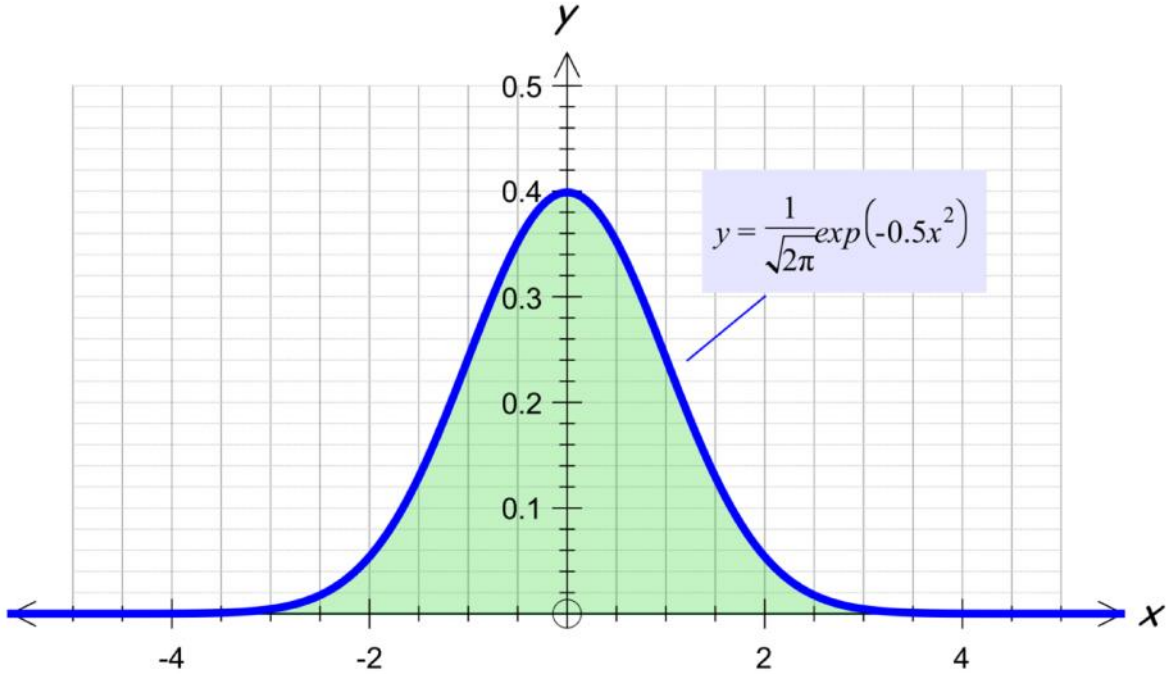
$$y = \frac{1}{x^2}$$

Discuss the discrepancy between the two answers.

Uses of Improper Integral

- Probability density functions, Fourier transforms, Laplace transforms, integral comparison test for infinite series
- Improper integrals are used all the time when calculating probabilities on the normal curve. In hypothesis testing we often ask what is the probability something is at least some value... this in essence is asking to take an integral from $[a,\infty]$.

Normal Distribution



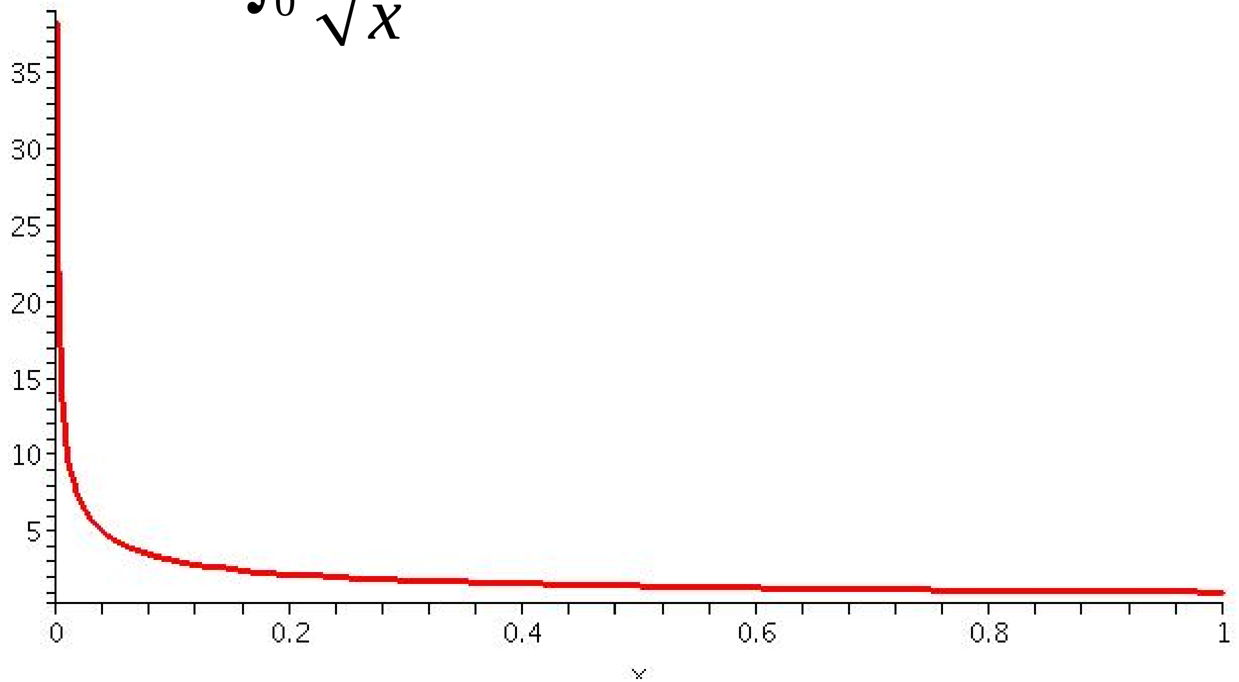
Normal Probability distribution have an area of 1 under the entire curve. The area under the curve is 1 as long as you are talking about the integral from negative infinity to positive infinity.

Applications of Improper Integral

- The most common application of such integrals is in probability and statistics, in which some quantity is modelled by a probability distribution which is supported on the entire real line, such as the normal distribution ("bell curve"). To determine, for instance, the probability that the quantity is above some fixed number N, requires the computation of the integral of the probability distribution function from N to positive infinity.

Example #1

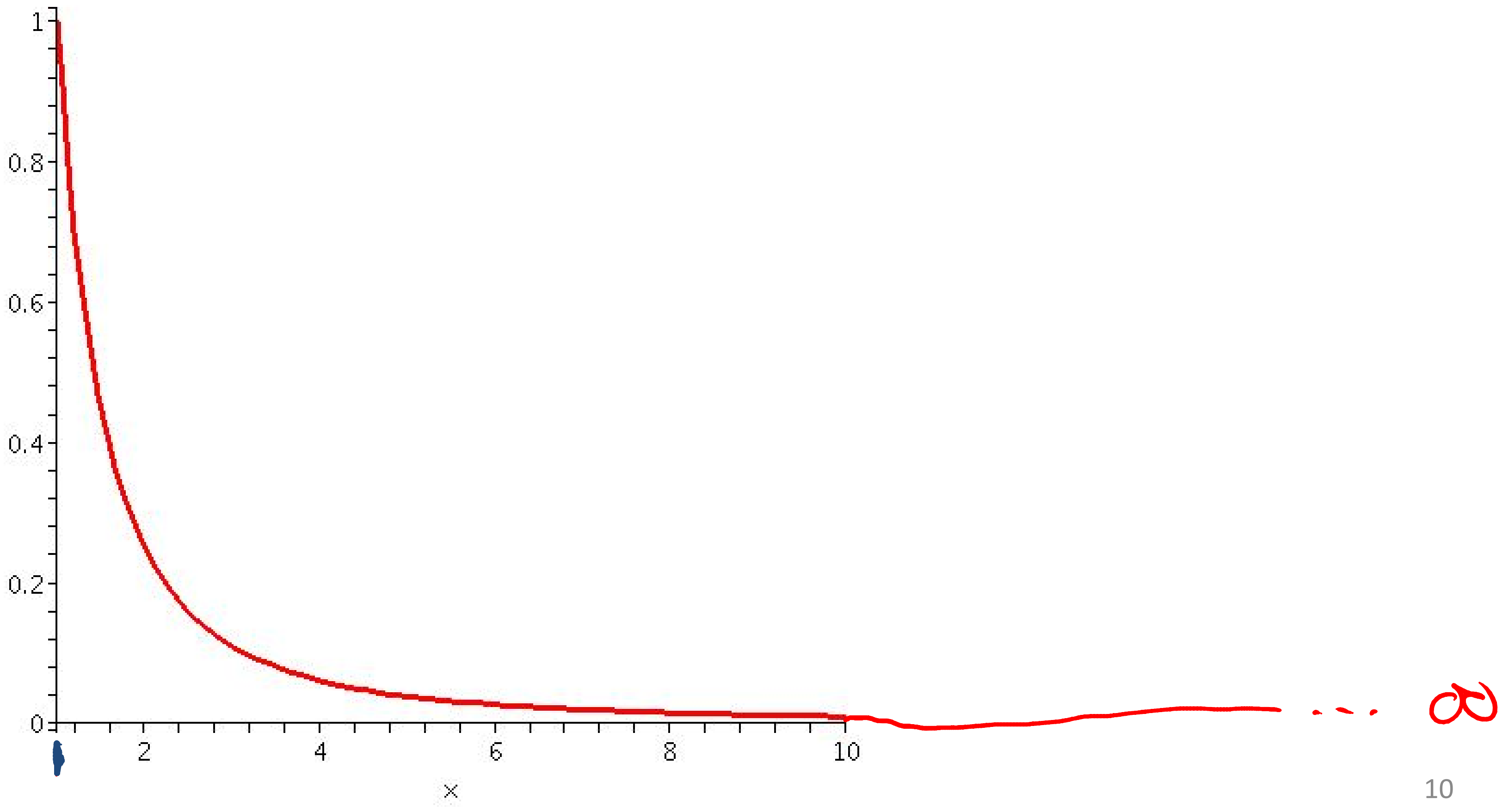
- The integral $\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x$ is improper…



… if you try to do this by drawing rectangles, you will run into problems!

Example #2

- The integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ is improper.



Limits

- If the range of the integral is finite but it has an infinity at one of its endpoints, we consider a limit. Remember our definition of integration:

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x = F(b) - F(a)$$

Limits

- If $f(x)$ has a singularity at a, then

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x = \lim_{X \to a}\left(F(b) - F(X)\right)$$

- If $f(x)$ has a singularity at b, then

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x = \lim_{X \to b}\left(F(X) - F(a)\right)$$

Example

- Consider our original example

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x$$

- Now we can find $F(x)$…

- So using

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x = \lim_{X \to a} F(b) - F(X)$$

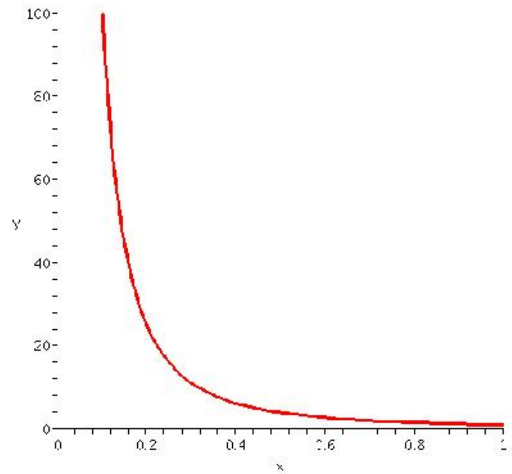
Example

- Although the integrand goes to infinity, the integral is finite!

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2$$

- But does this always work?

Example #2



- Consider the integral

$$\int_0^1 \frac{1}{x^2}\mathrm{d}x$$ where the integrand is infinite at $x=0$.

- First we compute the indefinite integral:

$$F(x) = \int f(x)\mathrm{d}x = \int x^{-2}\mathrm{d}x =$$

- Now take the limit: what happens?

$$\int_{x=a}^{x=b} f(x)\mathrm{d}x = \lim_{X \to a} F(b) - F(X)$$

Example #2

- For this curve, which also goes to infinity at $x=0$, the limit as x tends to zero is infinity, so the integral is infinite (or undefined).

- You may be looking at this and thinking 'just plug in the zero and see what the answer is', which would have worked in the last two cases, but doesn't always:

Example #3

- Consider

$$\int_0^1 \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) \mathrm{d}x$$

$\frac{\sin x}{x}$

which has terms that have singularities at $x=0$.

- Can you see what the indefinite integral ($F(x)$) is? (Hint – what is the integrand the derivative of?)

Example #3

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\int \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) \mathrm{d}x = \frac{\sin(x)}{x} + C$$

- So the definite integral is $\lim_{X \to 0} \left[\frac{\sin 1}{1} - \frac{\sin X}{X} \right]$
- What do you get here? What would you get just by subbing in $X=0$?

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\sin x \approx x$ for $x \ll 1$

Example #3

- So we find

$$\int_0^1 \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) \mathrm{d}x = \sin(1) - 1$$

The second kind of improper integral

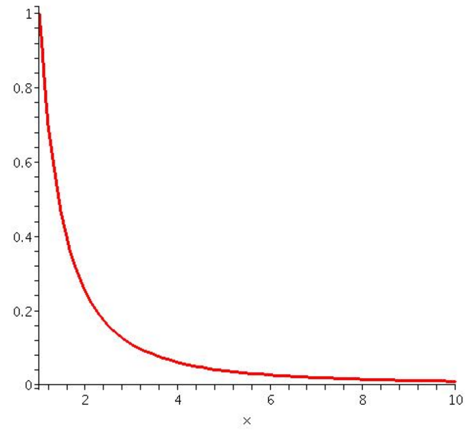
- These are integrals where one of the bounds of the integral is at infinity, like $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$
- We use limits here again. If the upper bound is infinite, we say

$$\int_{x=a}^{x=\infty} f(x)\mathrm{d}x = \lim_{X\to\infty}\Big(F(X) - F(a)\Big)$$

- If the lower bound is (-ve) infinite, then

$$\int_{x=-\infty}^{x=b} f(x)\mathrm{d}x = \lim_{X\to-\infty}\Big(F(b) - F(X)\Big)$$

Example #4



- So consider $\int_1^{\infty} \frac{1}{x^2} \mathrm{d}x$

- We can find the indefinite integral (as before):

$$F(x) = \int \frac{1}{x^2} \mathrm{d}x = -x^{-1} + C$$

- Using the limit formula,

$$\int_{x=a}^{x=\infty} f(x) \mathrm{d}x = \lim_{X \to \infty} F(X) - F(a)$$

- What is $\lim_{X \to \infty} \frac{1}{X}$?

Example #4

- Since $\lim_{X\to\infty}\frac{1}{X}=0$

$$\int_{x=1}^{x=\infty} f(x)\mathrm{d}x=\lim_{X\to\infty}F(X)-F(1)=0-(-1^{-1})=1$$

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x=1$$

- This infinite integral has a finite answer. But does this always work?

Example #5

- Consider the integral

$$\int_1^\infty x\,\mathrm{d}x$$

- The indefinite integral is trivial:
- Now use

$$\int_{x=a}^{x=\infty} f(x)\mathrm{d}x = \lim_{X\to\infty} F(X) - F(a)$$

… what do we get?

Example #5

- The integral

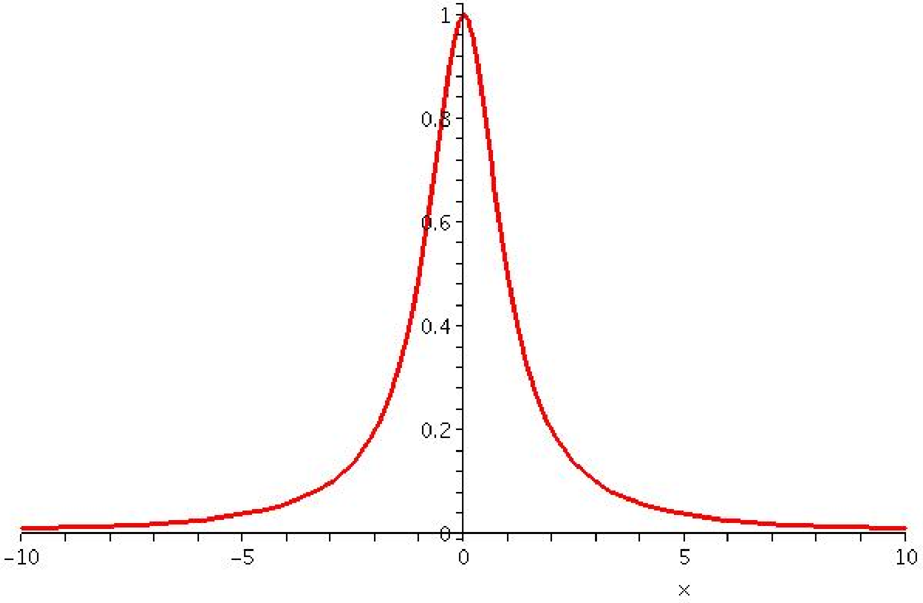
  $$\int_1^{\infty} x \, \mathrm{d}x$$

  is infinite (or undefined).
- In general we expect to run into problems unless the integrand tends to 0 as x tends to infinity.

Example #6

- Consider the doubly infinite integral

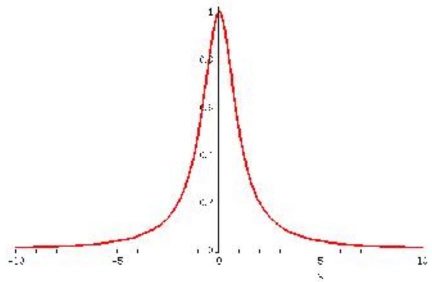
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \mathrm{d}x$$



- How can we tackle this?

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

Example #6



- It might be easiest to write this as

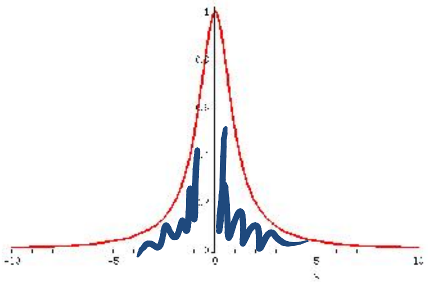
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_{0}^{+\infty} \frac{1}{1+x^2}\mathrm{d}x + \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x$$

- Then the first integral on the RHS is

$$\int_{x=0}^{x=\infty} f(x)\mathrm{d}x = \lim_{X\to\infty} F(X) - F(0) \quad \text{where} \quad F(x) = \arctan(x) + C$$

- The limit is $\pi/2$ (try it!) and $\arctan(0)=0$.

Example #6



- Writing our integral as

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \int_{0}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x$$

- We have that the first integral is $\frac{\pi}{2}$
- But what is the second integral (by symmetry, or by taking the limit again?) $\frac{\pi}{2}$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$$

Example #6

- So finally

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \text{®}$$

Your turn (time permitting)

- State whether the following improper integrals can be done, and, time permitting, do those that can.

$$\int_0^1 \frac{1}{x}\mathrm{d}x$$ ✗ $F(x) = \ln x$

$$\int_0^\infty \mathrm{d}x$$ ✗ $F(x) = x$

$$\int_0^1 \frac{1}{x^{1/3}}\mathrm{d}x$$ ✓

$$\int_0^\infty e^{-x}\mathrm{d}x$$ ✓ $F(x) = -e^{-x}$ $= 1$

$$\int_1^\infty \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right)\mathrm{d}x$$ $F(x) = \frac{\sin x}{x}$

Summary

- Improper integrals involve either the integrand, or the limits going to infinity.
- We can't do these by thinking about normal 'Riemann integration' where we break the integral up into a set of rectangles.
- However, we can solve them by taking limits.
- Improper integrals crop up in a wide variety of places, including physics, Fourier transforms, complex analysis… (more next year)

Homework

Exercise on Studynet,
and revision!
Past papers on
www.mathsforall.co.uk