MT1: Tutorial questions

23. Improper integration

1. State whether the following improper integrals have a defined value, and, if so, find that value.

   (a) $\displaystyle\int_0^\infty x^2 \mathrm{d}x$

   (b) $\displaystyle\int_0^1 \left(w^2 + w^{-1/4}\right) \mathrm{d}w$

   (c) $\displaystyle\int_0^1 \frac{1}{1-y} \mathrm{d}y$

   (d) $\displaystyle\int_0^\infty e^{-x} \mathrm{d}x$

   (e) $\displaystyle\int_0^\infty \sin(x) \mathrm{d}x$

   (f) $\displaystyle\int_0^\infty \frac{1}{q^2} \mathrm{d}q$

   (g) $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}} \mathrm{d}x$

2. The gravitational force between two bodies of mass m_1 and m_2 is given by

$$F = \frac{G m_1 m_2}{r^2}$$

   The work done against gravity in increasing the separation of the two bodies from $r = a$ to $r = b$ is given by

$$W = \int_a^b F \mathrm{d}r$$

   Suppose body 1 is the Earth, body 2 is a rocket; write down and solve an improper integral for the work done in moving the rocket from $r = R$ (the radius of the Earth) to $r = \infty$, giving your answer in terms of G, m_1, m_2 and R.

3. (a) Show that

$$\int \frac{1}{(x^2+2)^{3/2}} \mathrm{d}x = \frac{x}{2(x^2+2)^{1/2}} + C$$

   (Hint: use a trig substitution, $x = \sqrt{2}\tan\theta$, and note that this means that $\sin\theta = x/(x^2+2)^{1/2}$.)

   (b) Hence find

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+2)^{3/2}} \mathrm{d}x$$

4. For integer n, prove that

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n!$$

   (Hint: consider integration by parts. $n!$ is the factorial of n.)