Complete Solutions to Supplementary Exercises on Infinite Series

1. (a) We need to find the sum $\sum_{m=1}^{\infty}\frac{1}{(3m-2)(3m+1)}$. Converting the summand into partial fractions gives

$$\frac{1}{(3m-2)(3m+1)}=\frac{A}{3m-2}+\frac{B}{3m+1}$$

By the cover up rule we have

$$A=\frac{1}{\left(3(2/3)+1\right)}=\frac{1}{3}\ \text{ and }\ B=\frac{1}{\left(3(-1/3)-2\right)}=-\frac{1}{3}$$

Therefore

$$\frac{1}{(3m-2)(3m+1)}=\frac{1}{3(3m-2)}-\frac{1}{3(3m+1)}=\frac{1}{3}\left[\frac{1}{3m-2}-\frac{1}{3m+1}\right]$$

Let $S_n=\sum_{m=1}^{n}\frac{1}{(3m-2)(3m+1)}$ then by using these partial fractions we have:

$$\begin{aligned}
S_n&=\sum_{m=1}^{n}\frac{1}{3}\left[\frac{1}{3m-2}-\frac{1}{3m+1}\right]\\
&=\frac{1}{3}\sum_{m=1}^{n}\left[\frac{1}{3m-2}-\frac{1}{3m+1}\right]\\
&=\frac{1}{3}\left[\underbrace{\left(\frac{1}{3-2}-\frac{1}{3+1}\right)}_{m=1}+\underbrace{\left(\frac{1}{3(2)-2}-\frac{1}{3(2)+1}\right)}_{m=2}+\underbrace{\left(\frac{1}{3(3)-2}-\frac{1}{3(3)+1}\right)}_{m=3}+\cdots\right.\\
&\qquad\left.+\underbrace{\left(\frac{1}{3(n-1)-2}-\frac{1}{3(n-1)+1}\right)}_{m=n-1}+\underbrace{\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)}_{m=n}\right]\\
&=\frac{1}{3}\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-5}-\frac{1}{3n-2}\right)+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)\right]\\
&=\frac{1}{3}\left[1-\frac{1}{3n+1}\right]
\end{aligned}$$

The infinite series is given by

$$\begin{aligned}
\lim_{n\to\infty}\left(S_n\right)&=\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{(3m-2)(3m+1)}\right]\\
&=\lim_{n\to\infty}\left(\frac{1}{3}\left[1-\frac{1}{3n+1}\right]\right)=\frac{1}{3}
\end{aligned}$$

The given series converges with the sum $\frac{1}{3}$.

(b) We are given $\sum_{m=1}^{\infty}\left(\frac{3^m+2^m}{6^m}\right)$. Let S_n be the partial sum, then

$$S_n=\sum_{m=1}^{n}\left(\frac{3^m+2^m}{6^m}\right)=\sum_{m=1}^{n}\left[\left(\frac{3}{6}\right)^m+\left(\frac{2}{6}\right)^m\right]=\sum_{m=1}^{n}\left[\left(\frac{1}{2}\right)^m+\left(\frac{1}{3}\right)^m\right]$$

The infinite series is

$$\lim_{n\to\infty}\left(S_n\right)=\lim_{n\to\infty}\left(\sum_{m=1}^{n}\left[\left(\frac{1}{2}\right)^m+\left(\frac{1}{3}\right)^m\right]\right)\quad (*)$$

Note that each of these are geometric series with common ration $\frac{1}{2}$ and $\frac{1}{3}$:

$$\lim_{n\to\infty}\left(\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m\right)=\frac{1/2}{1-1/2}=1 \text{ and } \lim_{n\to\infty}\left(\sum_{m=1}^{n}\left(\frac{1}{3}\right)^m\right)=\frac{1/3}{1-1/3}=\frac{1}{2}$$

Substituting these into the above (*) gives

$$\lim_{n\to\infty}\left(S_n\right)=\lim_{n\to\infty}\left(\sum_{m=1}^{n}\left[\left(\frac{1}{2}\right)^m+\left(\frac{1}{3}\right)^m\right]\right)=1+\frac{1}{2}=\frac{3}{2}$$

(c) We need to find $\sum_{m=1}^{\infty}\frac{2m+1}{m^2\left(m+1\right)^2}$. Converting the summand into partial fractions gives

$$\frac{2m+1}{m^2\left(m+1\right)^2}=\frac{Am+B}{m^2}+\frac{C}{m+1}+\frac{D}{\left(m+1\right)^2}$$

Multiplying through by $m^2\left(m+1\right)^2$ yields

$$2m+1=\left(Am+B\right)\left(m+1\right)^2+Cm^2\left(m+1\right)+Dm^2 \qquad (\dagger)$$

Substituting $m=-1$ into (†) yields

$$2\left(-1\right)+1=0+0+D\left(-1\right)^2 \;\Rightarrow\; -1=D$$

Substituting $m=0$ into (†) gives

$$2\left(0\right)+1=\left(A\left(0\right)+B\right)\left(0+1\right)^2 \;\Rightarrow\; 1=B$$

Equating coefficients of m^3 in (†) gives

$$0=A+C \;\Rightarrow A=-C$$

Equating coefficients of m^2 in (†):

$$0 = 2A + B + C + D$$

Substituting the above evaluated $B = 1,\ D = -1$ into this gives

$$0 = 2A + 1 + C - 1 \ \Rightarrow 0 = 2A + C$$

From the two simultaneous equations involving A and C:

$$A = -C \ \text{ and } \ 0 = 2A + C \ \text{ we have } \ A = 0,\ \ C = 0$$

Therefore our partial fractions conversion is

$$\frac{2m+1}{m^2\left(m+1\right)^2} = \frac{1}{m^2} - \frac{1}{\left(m+1\right)^2}$$

The nth partial sum is given by

$$\begin{aligned}
S_n &= \sum_{m=1}^{n} \frac{2m+1}{m^2\left(m+1\right)^2} \\
&= \sum_{m=1}^{n} \left[\frac{1}{m^2} - \frac{1}{\left(m+1\right)^2}\right] \\
&= \left[\underbrace{\left(\frac{1}{1^2} - \frac{1}{\left(1+1\right)^2}\right)}_{m=1} + \underbrace{\left(\frac{1}{2^2} - \frac{1}{\left(2+1\right)^2}\right)}_{m=2} + \underbrace{\left(\frac{1}{3^2} - \frac{1}{\left(3+1\right)^2}\right)}_{m=2} + \cdots \right. \\
&\qquad \left. + \underbrace{\left(\frac{1}{\left(n-1\right)^2} - \frac{1}{\left(n-1+1\right)^2}\right)}_{m=n-1} + \underbrace{\left(\frac{1}{n^2} - \frac{1}{\left(n+1\right)^2}\right)}_{m=n}\right] \\
&= \left[\underbrace{\left(1 - \frac{1}{4}\right)}_{m=1} + \underbrace{\left(\frac{1}{4} - \frac{1}{9}\right)}_{m=2} + \underbrace{\left(\frac{1}{9} - \frac{1}{16}\right)}_{m=2} + \cdots + \underbrace{\left(\frac{1}{\left(n-1\right)^2} - \frac{1}{n^2}\right)}_{m=n-1} + \underbrace{\left(\frac{1}{n^2} - \frac{1}{\left(n+1\right)^2}\right)}_{m=n}\right] \\
&= 1 - \frac{1}{\left(n+1\right)^2}
\end{aligned}$$

The sum of the infinite series is given by

$$\lim_{n\to\infty}\left(S_n\right) = \lim_{n\to\infty}\left[1 - \frac{1}{\left(n+1\right)^2}\right] = 1$$

Therefore $\displaystyle\sum_{m=1}^{\infty} \frac{2m+1}{m^2\left(m+1\right)^2} = 1$.

(d) We need to find $\sum_{m=1}^{\infty} \frac{m}{(2m-1)^2(2m+1)^2}$. Converting the summand into partial fractions gives

$$\frac{m}{(2m-1)^2(2m+1)^2} = \frac{A}{2m-1} + \frac{B}{(2m-1)^2} + \frac{C}{2m+1} + \frac{D}{(2m+1)^2}$$

Multiplying this by $(2m-1)^2(2m+1)^2$ gives

$$m = A(2m-1)(2m+1)^2 + B(2m+1)^2 + C(2m-1)^2(2m+1) + D(2m-1)^2 \quad (*)$$

Substituting $m = \frac{1}{2}$ into (*) gives

$$\frac{1}{2} = A(0) + B\left(2\left(\frac{1}{2}\right)+1\right)^2 + C(0) + D(0) \quad \Rightarrow \quad \frac{1}{2} = 4B \quad \Rightarrow \quad \frac{1}{8} = B$$

Substituting $m = -\frac{1}{2}$ into (*) gives

$$-\frac{1}{2} = A(0) + B(0) + C(0) + D\left(2\left(-\frac{1}{2}\right)-1\right)^2 \Rightarrow -\frac{1}{2} = 4D \quad \Rightarrow \quad -\frac{1}{8} = D$$

Substituting $m = 0$ into (*) gives

$$0 = -A + B + C + D$$

Putting $B = \frac{1}{8}$ and $D = -\frac{1}{8}$ into this yields

$$0 = -A + \frac{1}{8} + C - \frac{1}{8} \quad \Rightarrow \quad A = C$$

Equating coefficients of m^3 in (*):

$$0 = 8A + 8C \quad \Rightarrow \quad A + C = 0$$

From the last two equations we have $A = 0$ and $C = 0$.

Substituting these evaluated A, B, C and D gives the partial decomposition:

$$\frac{m}{(2m-1)^2(2m+1)^2} = \frac{1}{8(2m-1)^2} - \frac{1}{8(2m+1)^2} = \frac{1}{8}\left[\frac{1}{(2m-1)^2} - \frac{1}{(2m+1)^2}\right]$$

The nth partial sum S_n is equal to

$$S_n = \sum_{m=1}^{n} \frac{m}{(2m-1)^2(2m+1)^2} = \frac{1}{8}\sum_{m=1}^{n}\left[\frac{1}{(2m-1)^2} - \frac{1}{(2m+1)^2}\right]$$

Expanding this out

$$S_n = \frac{1}{8}\sum_{m=1}^{n}\left[\frac{1}{(2m-1)^2} - \frac{1}{(2m+1)^2}\right]$$

$$= \frac{1}{8}\left[\underbrace{\left(\frac{1}{(2(1)-1)^2} - \frac{1}{(2(1)+1)^2}\right)}_{m=1} + \underbrace{\left(\frac{1}{(2(2)-1)^2} - \frac{1}{(2(2)+1)^2}\right)}_{m=2} + \underbrace{\left(\frac{1}{(2(3)-1)^2} - \frac{1}{(2(3)+1)^2}\right)}_{m=3} + \cdots + \underbrace{\left(\frac{1}{(2(n-1)-1)^2} - \frac{1}{(2(n-1)+1)^2}\right)}_{m=n-1} + \underbrace{\left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right)}_{m=n}\right]$$

$$= \frac{1}{8}\left[\underbrace{\left(\frac{1}{1} - \frac{1}{9}\right)}_{m=1} + \underbrace{\left(\frac{1}{9} - \frac{1}{25}\right)}_{m=2} + \underbrace{\left(\frac{1}{25} - \frac{1}{49}\right)}_{m=3} + \cdots + \underbrace{\left(\frac{1}{(2n-3)^2} - \frac{1}{(2n-1)^2}\right)}_{m=n-1} + \underbrace{\left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right)}_{m=n}\right]$$

$$= \frac{1}{8}\left[1 - \frac{1}{(2n+1)^2}\right]$$

The infinite sum is given by

$$\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left(\frac{1}{8}\left[1 - \frac{1}{(2n+1)^2}\right]\right) = \frac{1}{8}$$

(e) We are asked to find $\sum_{m=1}^{\infty}\tan^{-1}\left(\frac{1}{2m^2}\right)$. We first find the partial sum

$$\sum_{m=1}^{n}\tan^{-1}\left(\frac{1}{2m^2}\right)$$

In order to evaluate this partial sum we find the sum for $n = 2$:

$$\sum_{m=1}^{2}\tan^{-1}\left(\frac{1}{2m^2}\right) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{8}}{1-\left(\frac{1}{2}\times\frac{1}{8}\right)}\right) \quad [\text{By hint}]$$

$$= \tan^{-1}\left(\frac{10}{15}\right) = \tan^{-1}\left(\frac{2}{3}\right)$$

Now we find the sum for $n = 3$:

$$\sum_{m=1}^{3}\tan^{-1}\left(\frac{1}{2m^2}\right)=\sum_{m=1}^{2}\tan^{-1}\left(\frac{1}{2m^2}\right)+\tan^{-1}\left(\frac{1}{2\left(3^2\right)}\right)$$

$$=\underbrace{\tan^{-1}\left(\frac{2}{3}\right)}_{\text{by previous result}}+\tan^{-1}\left(\frac{1}{18}\right)$$

$$=\tan^{-1}\left(\frac{\frac{2}{3}+\frac{1}{18}}{1-\left(\frac{2}{3}\times\frac{1}{18}\right)}\right)=\tan^{-1}\left(\frac{39}{52}\right)=\tan^{-1}\left(\frac{3}{4}\right)$$

Similarly we find

$$\sum_{m=1}^{4}\tan^{-1}\left(\frac{1}{2m^2}\right)=\tan^{-1}\left(\frac{4}{5}\right)\text{ and}$$

$$\sum_{m=1}^{5}\tan^{-1}\left(\frac{1}{2m^2}\right)=\tan^{-1}\left(\frac{5}{6}\right)$$

Can you spot a pattern?

$$\sum_{m=1}^{n}\tan^{-1}\left(\frac{1}{2m^2}\right)=\tan^{-1}\left(\frac{n}{n+1}\right)$$

You can prove this by induction.

The infinite sum is given by

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\tan^{-1}\left(\frac{1}{2m^2}\right)\right]=\lim_{n\to\infty}\left[\tan^{-1}\left(\frac{n}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\left[\tan^{-1}\left(\frac{1}{1+1/n}\right)\right]=\tan^{-1}\left(1\right)=\frac{\pi}{4}$$

2. (a) We need to use the comparison test to find whether $\sum_{m=1}^{\infty}\frac{1}{\left(2m-1\right)2^{2m-1}}$ converges or not. Consider the summand:

$$\frac{1}{\left(2m-1\right)2^{2m-1}}<\frac{1}{2^{2m-1}}\leq\frac{1}{2^m}\qquad\left[\text{Because } 2m-1\geq m \text{ for } m\geq 1\right]$$

Now $\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m$ is a geometric series with the common ratio $r=\frac{1}{2}$ and since

$$\left|r\right|=\frac{1}{2}<1\text{ so }\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m\text{ converges.}$$

Hence by the comparison test $\sum_{m=1}^{\infty}\frac{1}{(2m-1)2^{2m-1}}$ converges.

(b) We need to test $\sum_{m=1}^{\infty}\sin\left(\frac{\pi}{2^m}\right)$ for convergence. The power series for sin is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

For $x > 0$ we have $\sin(x) < x$. Applying this to $\sin\left(\frac{\pi}{2^m}\right)$ gives

$$\sin\left(\frac{\pi}{2^m}\right) < \frac{\pi}{2^m}$$

Examining the infinite series $\sum_{m=1}^{\infty}\frac{\pi}{2^m}$ gives

$$\sum_{m=1}^{\infty}\frac{\pi}{2^m} = \pi\sum_{m=1}^{\infty}\frac{1}{2^m} = \pi\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m$$

This $\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m$ of course is a geometric series with common ration $r = \frac{1}{2} < 1$.

Since the common ratio is less than 1 so $\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m$ converges.

Hence by the comparison test we conclude that the given $\sum_{m=1}^{\infty}\sin\left(\frac{\pi}{2^m}\right)$ converges.

(c) We are asked to test $\sum_{m=1}^{\infty}\frac{m+1}{(m+2)m}$. Converting the summand into partial fractions:

$$\frac{m+1}{(m+2)m} = \frac{A}{m+2} + \frac{B}{m}$$

Cover up gives $A = \frac{-1}{-2} = \frac{1}{2}$ and $B = \frac{1}{2}$. Substituting these into the above gives

$$\frac{m+1}{(m+2)m} = \frac{1}{2}\left(\frac{1}{m+2}\right) + \frac{1}{2}\left(\frac{1}{m}\right)$$

From this we have $\frac{1}{2}\left(\frac{1}{m+2}\right) + \frac{1}{2}\left(\frac{1}{m}\right) > \frac{1}{2m}$. Now

$$\sum_{m=1}^{\infty} \frac{1}{2m} \text{ diverges}$$

By the comparison test we have $\sum_{m=1}^{\infty} \frac{m+1}{(m+2)m}$ diverges.

(d) *How do we test* $\sum_{m=1}^{\infty} \tan\left(\frac{\pi}{4m}\right)$ *for convergence?*

We need to find an inequality involving the tangent function. The power series expansion of tan is given in formula (7.18):

(7.18) $$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \quad \text{provided } |x| < \frac{\pi}{2}$$

From this we have $\tan(x) > x$ for $x \neq 0$. For the given summand we have

$$\tan\left(\frac{\pi}{4m}\right) > \frac{\pi}{4m}$$

The infinite series $\sum_{m=1}^{\infty} \frac{\pi}{4m}$ diverges so by the comparison test we conclude that $\sum_{m=1}^{\infty} \tan\left(\frac{\pi}{4m}\right)$ diverges.

(e) We need to test $\sum_{m=1}^{\infty} \frac{1}{m^2+1}$ for convergence. An inequality involving the summand is

$$\frac{1}{m^2+1} < \frac{1}{m^2} \quad \left[\text{Because } m^2+1 > m^2\right]$$

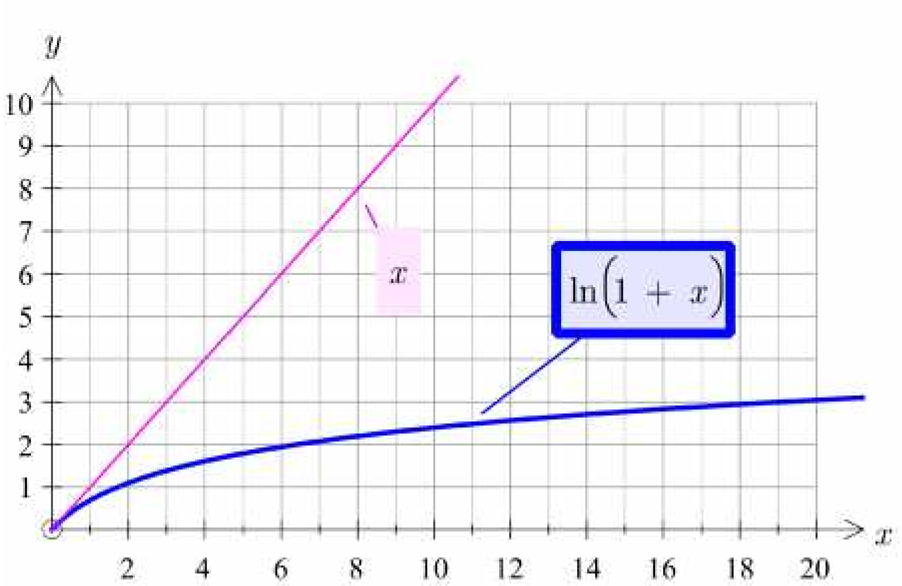
By the p – test we have $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges so by the comparison test $\sum_{m=1}^{\infty} \frac{1}{m^2+1}$ converges.

(f) We are given $\sum_{m=1}^{\infty} \frac{1}{3m-1}$. We have the inequality

$$3m - 1 < 3m \quad \Leftrightarrow \quad \frac{1}{3m-1} > \frac{1}{3m}$$

Also we know that $\sum_{m=1}^{\infty} \frac{1}{3m}$ diverges so by the comparison test $\sum_{m=1}^{\infty} \frac{1}{3m-1}$ diverges.

(g) We are asked to test $\sum_{m=1}^{\infty} \frac{1}{\ln(m+1)}$. Comparing the graphs of x and $\ln(1+x)$ gives:



From this we can see that $\ln(1+x) < x$ for $x \geq 1$. (It is a well – known identity.)

From this $\ln(1+x) < x$ we have $\dfrac{1}{\ln(1+x)} > \dfrac{1}{x}$ so

$$\frac{1}{\ln(1+m)} > \frac{1}{m}$$

The harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges so by the comparison test the given series $\sum_{m=1}^{\infty} \frac{1}{\ln(m+1)}$ diverges.

(h) *How do we test* $\sum_{m=1}^{\infty} \frac{1+m^2}{1+m^3}$ *for convergence?*

We first convert the summand into partial fractions:

$$\frac{1+m^2}{1+m^3} \underbrace{=}_{\text{Fatorising denominator}} \frac{m^2+1}{(m+1)(m^2-m+1)} = \frac{A}{m+1} + \frac{Bm+C}{m^2-m+1} \qquad (\dagger)$$

Multiplying both sides by $(m+1)(m^2-m+1)$ gives

$$m^2+1 = A(m^2-m+1) + (Bm+C)(m+1) \qquad (*)$$

Substituting $m=-1$ into (*) yields

$$(-1)^2+1 = A\left((-1)^2-(-1)+1\right) + (Bm+C)(0) \quad \Rightarrow \quad 2 = 3A \quad \Rightarrow \quad A = \frac{2}{3}$$

Substituting $m=0$ into (*) gives

$$1 = A + C \;\Rightarrow\; 1 = \frac{2}{3} + C \;\Rightarrow\; C = \frac{1}{3}$$

Equating coefficients of m^2 in (*) gives

$$1 = A + B \;\Rightarrow\; 1 = \frac{2}{3} + B \;\Rightarrow\; B = \frac{1}{3}$$

Substituting $A = \frac{2}{3}$, $B = \frac{1}{3}$ and $C = \frac{1}{3}$ into (†) gives

$$\frac{1+m^2}{1+m^3} = \frac{1}{3}\left[\frac{2}{m+1} + \frac{m+1}{m^2-m+1}\right] > \frac{1}{3}\left[\frac{2}{m+1}\right] = \frac{2}{3}\left[\frac{1}{m+1}\right]$$

The infinite series $\sum_{m=1}^{\infty}\frac{1}{m+1}$ diverges so $\sum_{m=1}^{\infty}\frac{2}{3}\left[\frac{1}{m+1}\right]$ diverges and so by the comparison test $\sum_{m=1}^{\infty}\frac{1+m^2}{1+m^3}$ diverges.

(i) We are asked to test $\sum_{m=1}^{\infty}\left(\frac{1+m^2}{1+m^3}\right)^2$ for convergence. Let us first consider the term that is being squared:

$$\frac{1+m^2}{1+m^3} < \frac{1+m^2}{m^3} \qquad \left[\text{Because } 1+m^3 > m^3\right]$$

Squaring this gives

$$\left(\frac{1+m^2}{m^3}\right)^2 = \frac{1+2m^2+m^4}{m^6} = \frac{1}{m^6} + \frac{2}{m^4} + \frac{1}{m^2}$$

Now each of these $\sum_{m=1}^{\infty}\frac{1}{m^6}$, $\sum_{m=1}^{\infty}\frac{2}{m^4}$ and $\sum_{m=1}^{\infty}\frac{1}{m^2}$ converge by the p – test. So

$$\sum_{m=1}^{\infty}\left(\frac{1+m^2}{m^3}\right)^2 = \sum_{m=1}^{\infty}\left(\frac{1}{m^6} + \frac{2}{m^4} + \frac{1}{m^2}\right) \text{ converges}$$

By the comparison test $\sum_{m=1}^{\infty}\left(\frac{1+m^2}{1+m^3}\right)^2$ converges.

(j) We are given $\sum_{m=1}^{\infty}\frac{1}{\sqrt{m^2+2m}}$. We have the inequality

$$m^2 + 2m < m^2 + 2m + 1 = (m+1)^2$$

Applying the inequality $x < y \;\Rightarrow\; \sqrt{x} < \sqrt{y}$ for $x > 0,\; y > 0$ we have

$$\sqrt{m^2+2m} < \sqrt{(m+1)^2} = m+1$$

Taking the reciprocal gives

$$\frac{1}{\sqrt{m^2+2m}} > \frac{1}{m+1}$$

The infinite series $\sum_{m=1}^{\infty}\frac{1}{m+1}$ diverges so by the comparison test we conclude the given series $\sum_{m=1}^{\infty}\frac{1}{\sqrt{m^2+2m}}$ diverges.

(k) We are asked to test $\sum_{m=1}^{\infty}\frac{\ln(m)}{\sqrt[4]{m^5}}$ for convergence. *How?*

From the well-known inequality:

$$\ln(x) < x \text{ for } x > 0$$

We have

$$\ln\left(x^{1/5}\right) < x^{1/5}$$

Applying the following logarithm law $\ln(x^n) = n\ln(x)$ to the left hand side:

$$\frac{1}{5}\ln(x) < x^{1/5} \quad\Leftrightarrow\quad \ln(x) < 5x^{1/5}$$

Using this on the numerator of the summand gives

$$\frac{\ln(m)}{\sqrt[4]{m^5}} < \frac{5m^{1/5}}{m^{5/4}} = \frac{5}{m^{\frac{5}{4}-\frac{1}{5}}} = \frac{5}{m^{\frac{21}{20}}}$$

Now by the $p-$test, $\sum_{m=1}^{\infty}\frac{5}{m^{21/20}}$ converges so by the comparison test the given series $\sum_{m=1}^{\infty}\frac{\ln(m)}{\sqrt[4]{m^5}}$ converges.

(l) We are asked to test $\sum_{m=1}^{\infty}\left(\sqrt{m}-\sqrt{m-1}\right)$. Consider the summand

$$\sqrt{m}-\sqrt{m-1} = \frac{\left(\sqrt{m}-\sqrt{m-1}\right)\left(\sqrt{m}+\sqrt{m-1}\right)}{\sqrt{m}+\sqrt{m-1}} \quad \left[\text{Multiplying by } \frac{\left(\sqrt{m}+\sqrt{m-1}\right)}{\sqrt{m}+\sqrt{m-1}} = 1\right]$$

$$= \frac{m-m+1}{\sqrt{m}+\sqrt{m-1}} = \frac{1}{\sqrt{m}+\sqrt{m-1}}$$

For the natural number m we have

$$m-1 < m \quad\Leftrightarrow\quad \sqrt{m-1} < \sqrt{m} \quad\Leftrightarrow\quad \sqrt{m}+\sqrt{m-1} < \sqrt{m}+\sqrt{m} = 2\sqrt{m}$$

Taking the reciprocal we have the inequality

$$\frac{1}{\sqrt{m}+\sqrt{m-1}} > \frac{1}{2\sqrt{m}}$$

By the p – test $\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}$ diverges so $\sum_{m=1}^{\infty}\frac{1}{2\sqrt{m}}$ diverges. Therefore by the comparison test we conclude that $\sum_{m=1}^{\infty}\left(\sqrt{m}-\sqrt{m-1}\right)$ diverges.

(m) We are asked to test $\sum_{m=1}^{\infty}\frac{1}{m}\left(\sqrt{m+1}-\sqrt{m-1}\right)$. Consider the summand:

$$\frac{1}{m}\left(\sqrt{m+1}-\sqrt{m-1}\right) = \frac{1}{m}\frac{\left(\sqrt{m+1}-\sqrt{m-1}\right)\left(\sqrt{m+1}+\sqrt{m-1}\right)}{\left(\sqrt{m+1}+\sqrt{m-1}\right)}$$

$$= \frac{1}{m}\frac{m+1-m+1}{\left(\sqrt{m+1}+\sqrt{m-1}\right)} = \frac{2}{m\left(\sqrt{m+1}+\sqrt{m-1}\right)}$$

Consider the bracket term of the denominator of the last expression:

$$\sqrt{m+1}+\sqrt{m-1} > \sqrt{m}+\sqrt{m-1} \geq \sqrt{m} \quad \text{for } m \geq 1$$

Substituting this inequality into the above gives

$$\frac{1}{m}\left(\sqrt{m+1}-\sqrt{m-1}\right) = \frac{2}{m\left(\sqrt{m+1}+\sqrt{m-1}\right)} < \frac{2}{m\left(\sqrt{m}\right)} = \frac{2}{m\left(m^{1/2}\right)} = \frac{2}{m^{3/2}}$$

By the p – test, $\sum_{m=1}^{\infty}\frac{1}{m^{3/2}}$ converges so $\sum_{m=1}^{\infty}\frac{2}{m^{3/2}}$ converges. Hence by the comparison test we conclude $\sum_{m=1}^{\infty}\frac{1}{m}\left(\sqrt{m+1}-\sqrt{m-1}\right)$ converges.

3. We apply the ratio test in each case for this question.

(a) We are asked to test $\sum_{m=1}^{\infty}\frac{1}{(2m+1)!}$ for convergence. Let $a_m = \frac{1}{(2m+1)!}$ then $a_{m+1} = \frac{1}{(2m+3)!}$. Using the ratio test $\lim_{m\to\infty}\left(\frac{a_{m+1}}{a_m}\right)$ gives

$$L = \lim_{m\to\infty}\left(\frac{1}{(2m+3)!} \div \frac{1}{(2m+1)!}\right)$$
$$= \lim_{m\to\infty}\left(\frac{(2m+1)!}{(2m+3)!}\right)$$
$$= \lim_{m\to\infty}\left(\frac{\cancel{(2m+1)!}}{(2m+3)(2m+2)\cancel{(2m+1)!}}\right) = \lim_{m\to\infty}\left(\frac{1}{(2m+3)(2m+1)}\right) = 0 < 1$$

Therefore the given series $\sum_{m=1}^{\infty}\frac{1}{(2m+1)!}$ converges

(b) Similarly for $\sum_{m=1}^{\infty}\frac{m}{2^m}$ we let $a_m = \frac{m}{2^m}$ then

$$a_{m+1} = \frac{m+1}{2^{m+1}}$$

Finding the limit;

$$L = \lim_{m\to\infty}\left(\frac{m+1}{2^{m+1}} \div \frac{m}{2^m}\right)$$
$$= \lim_{m\to\infty}\left(\frac{m+1}{2^{m+1}} \times \frac{2^m}{m}\right) \underbrace{=}_{\text{because } \frac{m+1}{m} = \frac{\cancel{m}}{\cancel{m}} + \frac{1}{m}} \frac{1}{2}\lim_{m\to\infty}\left(1+\frac{1}{m}\right) = \frac{1}{2}$$

By the ratio test $\sum_{m=1}^{\infty}\frac{m}{2^m}$ converges because $L = \frac{1}{2} < 1$.

(c) We are asked to test $\sum_{m=1}^{\infty} m\tan\left(\frac{\pi}{2^{m+1}}\right)$ for convergence. Let $a_m = m\tan\left(\frac{\pi}{2^{m+1}}\right)$ so $a_{m+1} = (m+1)\tan\left(\frac{\pi}{2^{m+2}}\right)$ and

$$L = \lim_{m\to\infty}\left[\frac{(m+1)\tan\left(\frac{\pi}{2^{m+2}}\right)}{m\tan\left(\frac{\pi}{2^{m+1}}\right)}\right] = \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)\frac{\tan\left(\frac{\pi}{2^{m+2}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)}\right]$$
$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)\lim_{m\to\infty}\left[\frac{\tan\left(\frac{\pi}{2^{m+2}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)}\right] \qquad (*)$$

Examining the fraction on the right hand side:

$$\frac{\tan\left(\frac{\pi}{2^{m+2}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)} = \frac{\tan\left(\frac{1}{2}\frac{\pi}{2^{m+1}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)} = \frac{\tan\left(\frac{x}{2}\right)}{\tan\left(x\right)} \quad \text{where} \quad x = \frac{\pi}{2^{m+1}}$$

Using the double angle formula of chapter 4:

$$(4.55) \qquad \tan\left(2A\right) = \frac{2\tan\left(A\right)}{1-\tan^2\left(A\right)}$$

On $\dfrac{\tan\left(\frac{x}{2}\right)}{\tan\left(x\right)}$ gives

$$\begin{aligned} \tan\left(\frac{x}{2}\right) \div \tan\left(x\right) &= \tan\left(x/2\right) \div \left(\frac{2\tan\left(x/2\right)}{1-\tan\left(x/2\right)}\right) \\ &= \tan\left(x/2\right) \times \left(\frac{1-\tan\left(x/2\right)}{2\tan\left(x/2\right)}\right) \\ &= \frac{1}{2}\left(1-\tan\left(x/2\right)\right) \end{aligned}$$

Substituting back $x = \dfrac{\pi}{2^{m+1}}$ gives

$$\frac{\tan\left(\frac{\pi}{2^{m+2}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)} = \frac{1}{2}\left(1-\tan\left(\frac{\pi}{2^{m+2}}\right)\right)$$

Evaluating the limit yields

$$\frac{1}{2}\lim_{m\to\infty}\left(1-\tan\left(\frac{\pi}{2^{m+2}}\right)\right) = \frac{1}{2}\left(1-0\right) = \frac{1}{2}$$

Putting this into (*) gives

$$L = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)\lim_{m\to\infty}\left[\frac{\tan\left(\frac{\pi}{2^{m+2}}\right)}{\tan\left(\frac{\pi}{2^{m+1}}\right)}\right] = 1\left(\frac{1}{2}\right) = \frac{1}{2}$$

Since $L = \dfrac{1}{2} < 1$ so by the ratio test $\displaystyle\sum_{m=1}^{\infty} m\tan\left(\frac{\pi}{2^{m+1}}\right)$ converges.

(d) We need to test $\displaystyle\sum_{m=1}^{\infty}\frac{2\times5\times\cdots\times\left(3m-1\right)}{1\times5\times\cdots\times\left(4m-3\right)}$. Let

$$a_m = \frac{2\times 5\times\cdots\times(3m-1)}{1\times 5\times\cdots\times(4m-3)} \text{ then } a_{m+1} = \frac{2\times 5\times\cdots\times(3m-1)\times(3m+2)}{1\times 5\times\cdots\times(4m-3)\times(4m+1)}$$

Evaluating the limit

$$L = \lim_{m\to\infty}\left(a_{m+1} \div a_m\right) =$$

$$= \lim_{m\to\infty}\left(\frac{2\times 5\times\cdots\times(3m-1)\times(3m+2)}{1\times 5\times\cdots\times(4m-3)\times(4m+1)} \div \frac{2\times 5\times\cdots\times(3m-1)}{1\times 5\times\cdots\times(4m-3)}\right)$$

$$= \lim_{m\to\infty}\left(\frac{\cancel{2}\times\cancel{5}\times\cdots\times\cancel{(3m-1)}\times(3m+2)}{\cancel{1}\times\cancel{5}\times\cdots\times\cancel{(4m-3)}\times(4m+1)} \times \frac{\cancel{1}\times\cancel{5}\times\cdots\times\cancel{(4m-3)}}{\cancel{2}\times\cancel{5}\times\cdots\times\cancel{(3m-1)}}\right)$$

$$= \lim_{m\to\infty}\left(\frac{3m+2}{4m+1}\right) = \lim_{m\to\infty}\left(\frac{3+\frac{2}{m}}{4+\frac{1}{m}}\right) = \frac{3}{4} < 1$$

Since $L < 1$ so the given series $\sum_{m=1}^{\infty}\frac{2\times 5\times\cdots\times(3m-1)}{1\times 5\times\cdots\times(4m-3)}$ converges.

(e) We have to test $\sum_{m=1}^{\infty}\frac{m^2}{3^m}$ for convergence. Let

$$a_m = \frac{m^2}{3^m} \text{ then } a_{m+1} = \frac{(m+1)^2}{3^{m+1}}$$

Evaluating the limit

$$L = \lim_{m\to\infty}\left[\frac{(m+1)^2}{3^{m+1}} \div \frac{m^2}{3^m}\right]$$

$$= \lim_{m\to\infty}\left[\frac{(m+1)^2}{3^{m+1}} \times \frac{3^m}{m^2}\right]$$

$$= \frac{1}{3}\lim_{m\to\infty}\left(\frac{m+1}{m}\right)^2 = \frac{1}{3}\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^2 = \frac{1}{3}$$

Since $L = \frac{1}{3} < 1$ so the given series $\sum_{m=1}^{\infty}\frac{m^2}{3^m}$ converges.

(f) We are given the series $\sum_{m=1}^{\infty}\frac{1\times 3\times\cdots\times(2m-1)}{3^m\times m!}$. Let

$$a_m = \frac{1\times 3\times\cdots\times(2m-1)}{3^m\times m!} \text{ then } a_{m+1} = \frac{1\times 3\times\cdots\times(2m-1)\times(2m+1)}{3^{m+1}\times(m+1)!}$$

Working out the limit gives

$$L=\lim_{m\to\infty}\left(\frac{a_{m+1}}{a_m}\right)=\lim_{m\to\infty}\left[\frac{1\times3\times\cdots\times(2m-1)\times(2m+1)}{3^{m+1}\times(m+1)!}\div\frac{1\times3\times\cdots\times(2m-1)}{3^m\times m!}\right]$$

$$=\lim_{m\to\infty}\left[\frac{\cancel{1}\times\cancel{3}\times\cdots\times\cancel{(2m-1)}\times(2m+1)}{3^{\cancel{m}+1}\times(m+1)\times\cancel{(m!)}}\times\frac{\cancel{3^m}\times\cancel{(m!)}}{\cancel{1}\times\cancel{3}\times\cdots\times\cancel{(2m-1)}}\right]$$

$$=\frac{1}{3}\lim_{m\to\infty}\left[\frac{2m+1}{m+1}\right]=\frac{1}{3}\lim_{m\to\infty}\left[\frac{2+\frac{1}{m}}{1+\frac{1}{m}}\right]=\frac{1}{3}(2)=\frac{2}{3}$$

Since $L=\frac{2}{3}<1$ so the given series $\sum_{m=1}^{\infty}\frac{1\times3\times\cdots\times(2m-1)}{3^m\times m!}$ converges.

(g) We are asked to test $\sum_{m=1}^{\infty}\frac{m}{(m+1)!}$. Let

$$a_m=\frac{m}{(m+1)!}\quad\text{then}\quad a_{m+1}=\frac{m+1}{(m+2)!}$$

Finding the limit

$$L=\lim_{m\to\infty}\left[\frac{m+1}{(m+2)!}\times\frac{(m+1)!}{m}\right]$$

$$=\lim_{m\to\infty}\left[\frac{m+1}{m(m+2)}\right]\qquad\left[\text{Because }\frac{(m+1)!}{(m+2)!}=\frac{\cancel{(m+1)!}}{(m+2)\cancel{(m+1)!}}\right]$$

$$=\lim_{m\to\infty}\left[\frac{m+1}{m^2+2m}\right]\underbrace{=}_{\substack{\text{Dividing}\\\text{numerator and denominator}\\\text{by }m^2}}\lim_{m\to\infty}\left[\frac{\frac{1}{m}+\frac{1}{m^2}}{1+\frac{2}{m}}\right]=0$$

Since $L=0<1$ we conclude by the ratio test, $\sum_{m=1}^{\infty}\frac{m}{(m+1)!}$ converges.

(h) We are asked to test $\sum_{m=1}^{\infty}\frac{(m+1)!}{2^m m!}$. Let

$$a_m=\frac{(m+1)!}{2^m m!}\quad\text{then}\quad a_{m+1}=\frac{(m+2)!}{2^{m+1}(m+1)!}$$

Evaluating the limiting ratio

$$L = \lim_{m\to\infty}\left[\frac{(m+2)!}{2^{m+1}(m+1)!}\times\frac{2^m m!}{(m+1)!}\right]$$

$$= \frac{1}{2}\lim_{m\to\infty}\left[\frac{m+2}{m+1}\right] \qquad \left[\begin{array}{l}\text{Because} \\ \dfrac{(m+2)!}{(m+1)!}\times\dfrac{m!}{(m+1)!} = \dfrac{(m+2)\cancel{(m+1)!}}{\cancel{(m+1)!}}\times\dfrac{\cancel{m!}}{(m+1)\cancel{m!}}\end{array}\right]$$

$$= \frac{1}{2}\lim_{m\to\infty}\left[\frac{1+\frac{2}{m}}{1+\frac{1}{m}}\right] = \frac{1}{2}(1) = \frac{1}{2}$$

Hence by the ratio test the given series $\sum_{m=1}^{\infty}\frac{(m+1)!}{2^m m!}$ converges.

(i) Again using the ratio test to see whether $\sum_{m=1}^{\infty} m^2 \sin\left(\frac{\pi}{2^m}\right)$ converges or not.

Let $a_m = m^2\sin\left(\frac{\pi}{2^m}\right)$ then $a_{m+1} = (m+1)^2\sin\left(\frac{\pi}{2^{m+1}}\right)$. We have

$$L = \lim_{m\to\infty}\left[\frac{(m+1)^2\sin\left(\frac{\pi}{2^{m+1}}\right)}{m^2\sin\left(\frac{\pi}{2^m}\right)}\right]$$

$$= \lim_{m\to\infty}\left[\left(\frac{m+1}{m}\right)^2\frac{\sin\left(\frac{1}{2}\frac{\pi}{2^m}\right)}{\sin\left(\frac{\pi}{2^m}\right)}\right] = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^2\lim_{m\to\infty}\left[\frac{\sin\left(\frac{1}{2}x\right)}{\sin(x)}\right] \quad \text{where } x = \frac{\pi}{2^m}$$

Now using the trigonometric identity

(4.53) $\qquad \sin(2A) = 2\sin(A)\cos(A)$

From this we have $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$. Substituting this into the above on the extreme right hand side term gives

$$\frac{\sin\left(\frac{1}{2}x\right)}{\sin(x)} = \frac{\cancel{\sin\left(\frac{x}{2}\right)}}{2\cancel{\sin\left(\frac{x}{2}\right)}\cos\left(\frac{x}{2}\right)} = \frac{1}{2\cos\left(\frac{x}{2}\right)}$$

Substituting back $x = \frac{\pi}{2^m}$ and evaluating the limit on the right in the above

$$\lim_{m\to\infty}\left[\frac{\sin\left(\frac{1}{2}x\right)}{\sin\left(x\right)}\right]=\lim_{m\to\infty}\left(\frac{1}{2\cos\left(\frac{x}{2}\right)}\right)$$

$$=\lim_{m\to\infty}\left(\frac{1}{2\cos\left(\frac{\pi}{2^{m+1}}\right)}\right)$$

$$=\frac{1}{2}\qquad\left[\text{Because }\lim_{m\to\infty}\left(\frac{\pi}{2^{m+1}}\right)=0\text{ so }\lim_{m\to\infty}\left[\cos\left(\frac{\pi}{2^{m+1}}\right)\right]=\cos\left(0\right)=1\right]$$

Substituting this $\lim_{m\to\infty}\left[\frac{\sin\left(\frac{1}{2}x\right)}{\sin\left(x\right)}\right]=\frac{1}{2}$ into the above gives

$$L=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^2\lim_{m\to\infty}\left[\frac{\sin\left(\frac{1}{2}x\right)}{\sin\left(x\right)}\right]=1\times\frac{1}{2}=\frac{1}{2}$$

By the ratio test we conclude that $\sum_{m=1}^{\infty}m^2\sin\left(\frac{\pi}{2^m}\right)$ converges.

4. In this question we use the integral test for convergence or non – convergence. Since we will be dealing with a definite integral (improper integral) so we will ignore the constant of integration in our working.

(a) We are given the series $\sum_{m=1}^{\infty}\frac{1}{\left(m+1\right)\ln^2\left(m+1\right)}$. Let

$$f\left(x\right)=\frac{1}{\left(x+1\right)\ln^2\left(x+1\right)}=\frac{1}{\left(x+1\right)\left[\ln\left(x+1\right)\right]^2}$$

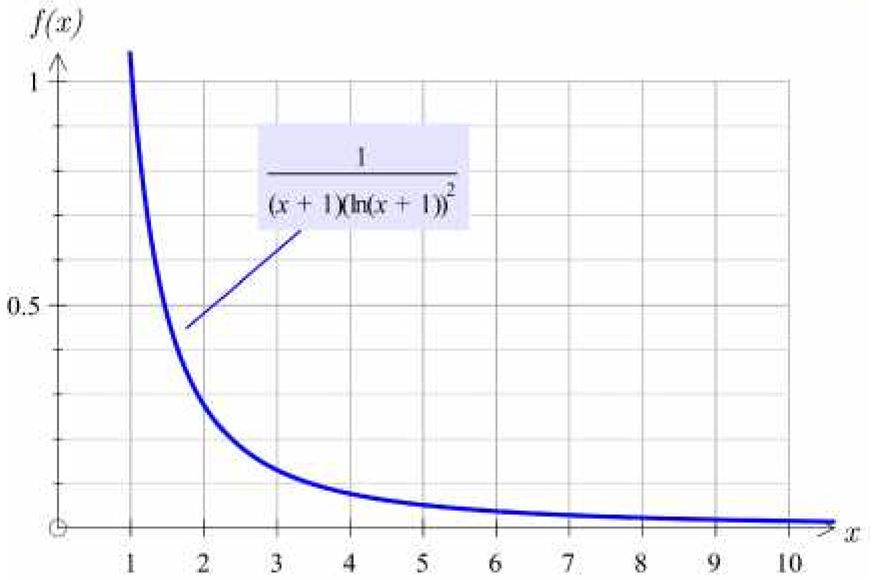
Then for $x\geq 1$ we have

(i) $f\left(x\right)>0$ (Positive) because $\left(x+1\right)$ and $\ln^2\left(x+1\right)$ are both positive.

(ii) $f\left(x\right)$ is *decreasing* because $\left(x+1\right)\left[\ln\left(x+1\right)\right]^2$ is *increasing*.

(iii) $f\left(x\right)$ is continuous because $\left(x+1\right)$ and $\ln^2\left(x+1\right)$ are both continuous therefore $\left(x+1\right)\ln^2\left(x+1\right)$ is continuous which implies $f\left(x\right)$ is continuous.

The graph of this function is given by:



We can now apply the integral test because *all* three conditions are satisfied.

Consider the improper integral

$$\int_1^\infty \frac{1}{\left(x+1\right)\left[\ln\left(x+1\right)\right]^2}\mathrm{d}x = \lim_{M\to\infty}\int_1^M \frac{\mathrm{d}x}{\left(x+1\right)\left[\ln\left(x+1\right)\right]^2}$$

How do we integrate this function?

By substitution. Let $u = \ln\left(x+1\right)$ then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x+1} \quad \Rightarrow \quad \mathrm{d}x = \left(x+1\right)\mathrm{d}u$$

First finding the indefinite integral we have

$$\int \frac{\mathrm{d}x}{\left(x+1\right)\left[\ln\left(x+1\right)\right]^2} = \int \frac{\cancel{\left(x+1\right)}\mathrm{d}u}{\cancel{\left(x+1\right)}u^2}$$

$$= \int u^{-2}\mathrm{d}u = -u^{-1} + C = -\frac{1}{\ln\left(x+1\right)} + C$$

Working out the improper integral

$$\lim_{M\to\infty}\int_1^M \frac{\mathrm{d}x}{\left(x+1\right)\left[\ln\left(x+1\right)\right]^2} = \lim_{M\to\infty}\left[-\frac{1}{\ln\left(x+1\right)}\right]_1^M$$

$$= \lim_{M\to\infty}\left[\frac{1}{\ln\left(1+1\right)} - \frac{1}{\ln\left(M+1\right)}\right] = \frac{1}{\ln\left(2\right)}$$

Since the improper integral converges so by the integral test the given series $\sum_{m=1}^{\infty}\frac{1}{(m+1)\ln^2(m+1)}$ converges.

(b) Now we are asked to use the integral test on the series $\sum_{m=2}^{\infty}\frac{1}{m\ln(m)}$.

Let $f(x)=\frac{1}{x\ln(x)}$ then for $x\geq 2$ we have

(i) $f(x)>0$ (Positive) because $x>0$ and $\ln(x)>0$ for $x\geq 2$.

(ii) In order to use the integral test we need to check that $f(x)$ is decreasing. We find the derivative of $f(x)$:

$$f'(x)=\left[\frac{1}{x\ln(x)}\right]'=\frac{0-\left(\ln(x)+\frac{x}{x}\right)}{\left(x\ln(x)\right)^2}=-\left[\frac{\ln(x)+1}{\left(x\ln(x)\right)^2}\right]$$

For $x\geq 2$ we have $\ln(x)+1>0$ and $\left(x\ln(x)\right)^2>0$ so

$$f'(x)=-\left[\frac{\ln(x)+1}{\left(x\ln(x)\right)^2}\right]<0$$

Therefore $f(x)$ is a strictly decreasing function.

(iii) $f(x)$ is continuous because x and $\ln(x)$ are continuous for $x\geq 2$ and so $f(x)=\frac{1}{x\ln(x)}$ is continuous.

As *all* three conditions are satisfied so we can apply the integral test.

Consider the improper integral:

$$\int_2^{\infty}\frac{1}{x\ln(x)}\mathrm{d}x=\lim_{M\to\infty}\int_2^{M}\frac{1}{x\ln(x)}\mathrm{d}x$$

Again we use integration by substitution with $u=\ln(x)$. Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}\quad\Rightarrow\quad \mathrm{d}x=x\mathrm{d}u$$

Changing the limits;

When $x=M$ then $u=\ln(M)$ and when $x=2$ then $u=\ln(2)$.

We have

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{M\to\infty}\int_{\ln(2)}^{\ln(M)} \frac{1}{\cancel{x}u}\,\cancel{x}du$$

$$= \lim_{M\to\infty}\int_{\ln(2)}^{\ln(M)} \frac{du}{u} = \lim_{M\to\infty}\Big[\ln(u)\Big]_{\ln(2)}^{\ln(M)}$$

$$= \lim_{M\to\infty}\Big[\ln\big(\ln(M)\big) - \ln\big(\ln(2)\big)\Big] = +\infty$$

This improper integral diverges so by the integral test we conclude that the given series $\sum_{m=2}^{\infty}\frac{1}{m\ln(m)}$ diverges.

(c) We are asked to test the series $\sum_{m=1}^{\infty}\left(\frac{1+m}{1+m^2}\right)^2$. Let

$$f(x) = \left(\frac{1+x}{1+x^2}\right)^2$$

For $x \ge 1$ we have

(i) $f(x) > 0$ [Positive] because we have a square function.

(ii) Finding the derivative of $f(x)$ we have

$$f'(x) = \left[\left(\frac{1+x}{1+x^2}\right)^2\right]' = 2\left(\frac{1+x}{1+x^2}\right)\left(\frac{1+x^2-2x(1+x)}{(1+x^2)^2}\right)$$

$$= 2\left(\frac{1+x}{1+x^2}\right)\left(\frac{1-2x-x^2}{(1+x^2)^2}\right)$$

$$= -2\left(\frac{1+x}{1+x^2}\right)\left(\frac{x^2+2x-1}{(1+x^2)^2}\right)$$

Now for $x \ge 1$ the numerator x^2+2x-1 is positive. All the other terms apart from -2 in the above are positive so $f'(x) < 0$ which implies that $f(x)$ is a strictly decreasing function.

(iii) Since $1+x$ and $1+x^2$ are continuous functions so

$$\frac{1+x}{1+x^2} \text{ is continuous implies } f(x) = \left(\frac{1+x}{1+x^2}\right)^2 \text{ is continuous}$$

All three conditions of the integral test are satisfied. Consider the improper integral

$$\int_1^\infty \left(\frac{1+x}{1+x^2}\right)^2 \mathrm{d}x = \lim_{M\to\infty} \int_1^M \left(\frac{1+x}{1+x^2}\right)^2 \mathrm{d}x$$

Let us first examine the indefinite integral:

$$\int \left(\frac{1+x}{1+x^2}\right)^2 \mathrm{d}x = \int \frac{x^2+2x+1}{\left(1+x^2\right)^2} \mathrm{d}x$$

Converting the integrand into partial fractions gives

$$\frac{x^2+2x+1}{\left(1+x^2\right)^2} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{\left(1+x^2\right)^2} \qquad (\dagger)$$

Multiplying through by $\left(1+x^2\right)^2$ gives

$$x^2+2x+1 = \left(Ax+B\right)\left(1+x^2\right) + Cx + D \qquad (*)$$

Equating coefficients of

x^3: $\quad 0 = A$

x^2: $\quad 1 = B$

x: $\quad 2 = A + C = 0 + C = C$

Const: $\quad 1 = B + D = 1 + D \quad \Rightarrow \quad D = 0$

Substituting these values $A=0,\ B=1,\ C=2$ and $D=0$ into $(\dagger)$:

$$\frac{x^2+2x+1}{\left(1+x^2\right)^2} = \frac{1}{1+x^2} + \frac{2x}{\left(1+x^2\right)^2}$$

Integrating this yields

$$\int \frac{x^2+2x+1}{\left(1+x^2\right)^2} \mathrm{d}x = \int \frac{1}{1+x^2} \mathrm{d}x + \int \frac{2x}{\left(1+x^2\right)^2} \mathrm{d}x$$

$$= \underbrace{\tan^{-1}\left(x\right)}_{\text{standard integral (8.26)}} + \underbrace{\int \frac{\cancel{2x}}{u^2} \frac{\mathrm{d}u}{\cancel{2x}}}_{\text{substituting } 1+x^2=u}$$

$$= \tan^{-1}\left(x\right) - u^{-1} = \tan^{-1}\left(x\right) - \frac{1}{1+x^2}$$

Now evaluating the improper integral from above

$$\lim_{M\to\infty}\int_1^M\left(\frac{1+x}{1+x^2}\right)^2 dx = \lim_{M\to\infty}\left[\tan^{-1}(x)-\frac{1}{1+x^2}\right]_1^M$$
$$= \lim_{M\to\infty}\left\{\left[\tan^{-1}(M)-\frac{1}{1+M^2}\right]-\left[\tan^{-1}(1)-\frac{1}{2}\right]\right\}$$
$$= \frac{\pi}{2}-0-\frac{\pi}{4}+\frac{1}{2}=\frac{\pi}{4}+\frac{1}{2}$$

The improper integral converges therefore by the integral test we conclude that the given series $\sum_{m=1}^{\infty}\left(\frac{1+m}{1+m^2}\right)^2$ converges.

(d) We need to test $\sum_{m=2}^{\infty}\frac{1}{\sqrt{m}}\ln\left(\frac{m+1}{m-1}\right)$ for convergence.

Let $f(x)=\frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)$. For $x\geq 2$ we have

(i) $f(x)>0$ [Positive] because for $x\geq 2$ the logarithmic argument $\frac{x+1}{x-1}>1$ so the natural log of this is positive and of course $\sqrt{x}$ is positive.

(ii) To find the derivative of $f(x)$ we rewrite $f(x)$ by using the laws of logs as

$$f(x)=\frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)=\frac{\left[\ln(x+1)-\ln(x-1)\right]}{\sqrt{x}}$$

Applying the quotient rule we have

$$f'(x)=\frac{\left[\frac{1}{x+1}-\frac{1}{x-1}\right]\sqrt{x}-\frac{1}{2\sqrt{x}}\left[\ln(x+1)-\ln(x-1)\right]}{x}$$

$$\underbrace{=}_{\substack{\text{Multiplying}\\ \text{numerator and denominator}\\ \text{by } 2\sqrt{x}}}\frac{2\left[\frac{-2}{x^2-1}\right]-\left[\ln\left(\frac{x+1}{x-1}\right)\right]}{2x\sqrt{x}}=-\frac{\left[\frac{4}{x^2-1}\right]+\left[\ln\left(\frac{x+1}{x-1}\right)\right]}{2x\sqrt{x}}$$

For $x\geq 2$ we have $f'(x)<0$ because of negative sign in front of the fraction and the fraction is positive. Hence $f(x)$ is strictly decreasing.

(iii) We have $\frac{1}{\sqrt{x}}$ is continuous and $\ln\left(\frac{x+1}{x-1}\right)$ is continuous therefore

$f(x)=\frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)$ is continuous.

We can now apply the integral test.

We need to test whether $\int_2^\infty \frac{1}{\sqrt{x}} \ln\left(\frac{x+1}{x-1}\right) \mathrm{d}x$ converges or not. Let us first examine the indefinite integral:

$$\int \frac{1}{\sqrt{x}} \ln\left(\frac{x+1}{x-1}\right) \mathrm{d}x$$

How do we integrate this function?

By parts. Let $u = \ln\left(\frac{x+1}{x-1}\right)$ and $v' = \frac{1}{\sqrt{x}} = x^{-1/2}$. Differentiating one and integrating the other gives

$$u' = \frac{2}{1-x^2} \qquad \left[\text{By (8.30) } \int \frac{1}{1-u^2} = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| \text{ in reverse}\right]$$

$$v = \int x^{-1/2}\mathrm{d}x = 2x^{1/2} = 2\sqrt{x}$$

Applying the integration by parts formula gives

$$\begin{aligned}\int \frac{1}{\sqrt{x}} \ln\left(\frac{x+1}{x-1}\right) \mathrm{d}x &= uv - \int (u'v)\mathrm{d}x \\ &= 2\sqrt{x}\ln\left(\frac{x+1}{x-1}\right) - 4\int \frac{\sqrt{x}}{1-x^2}\mathrm{d}x \\ &= 2\sqrt{x}\ln\left(\frac{x+1}{x-1}\right) + 4\int \frac{\sqrt{x}}{x^2-1}\mathrm{d}x \qquad (*)\end{aligned}$$

We need to find the integral on the right hand side of (*). *How?*

Use integration by substitution with $p = \sqrt{x}$. Then

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad \mathrm{d}x = 2\sqrt{x}\mathrm{d}p = 2p\mathrm{d}p$$

Therefore

$$\begin{aligned}\int \frac{\sqrt{x}}{x^2-1}\mathrm{d}x &= \int \frac{p}{p^4-1}2p\ \mathrm{d}p \\ &= 2\int \frac{p^2}{p^4-1}\ \mathrm{d}p \qquad (**)\end{aligned}$$

Converting the integrand in (**) to partial fractions we have

$$\frac{p^2}{p^4-1} = \frac{p^2}{(p^2-1)(p^2+1)} = \frac{Ap+B}{p^2-1} + \frac{Cp+D}{p^2+1} \qquad (\dagger)$$

Multiplying both sides by $(p^2-1)(p^2+1)$ gives

$$p^2 = \left(Ap + B\right)\left(p^2 + 1\right) + \left(Cp + D\right)\left(p^2 - 1\right)$$

Substituting $p = 1$ into this gives

$$1 = 2\left(A + B\right) = 2A + 2B$$

Equating coefficients of

p^3: $\quad 0 = A + C \;\Rightarrow\; A = -C$

p^2: $\quad 1 = B + D$

p: $\quad 0 = A - C \;\Rightarrow\; A = C$

From the p^3 and p coefficients we have $A = C = 0$.

Substituting $A = 0$ into first equation gives

$$1 = 2B \;\Rightarrow\; B = \frac{1}{2}$$

Substituting $B = \frac{1}{2}$ into the above coefficients of p^2 gives

$$1 = \frac{1}{2} + D \;\Rightarrow\; D = \frac{1}{2}$$

Putting these $A = C = 0$, $B = \frac{1}{2}$ and $D = \frac{1}{2}$ into (†):

$$\frac{p^2}{\left(p^2 - 1\right)\left(p^2 + 1\right)} = \frac{1}{2}\left(\frac{1}{p^2 - 1} + \frac{1}{p^2 + 1}\right)$$

Substituting this result into (**) gives

$$\begin{aligned}\int \frac{\sqrt{x}}{1 - x^2}\mathrm{d}x &= 2\int \frac{p^2}{p^4 - 1}\,\mathrm{d}p \\ &= \int \left(\frac{1}{p^2 - 1} + \frac{1}{p^2 + 1}\right)\mathrm{d}p \\ &= \int \frac{1}{\left(p - 1\right)\left(p + 1\right)}\mathrm{d}p + \underbrace{\tan^{-1}\left(p\right)}_{\text{by (8.26)}}\end{aligned}$$

The integral on the right hand side can be written as

$$\begin{aligned}\int \frac{1}{\left(p - 1\right)\left(p + 1\right)}\mathrm{d}p &= \int \left(\frac{1}{2}\frac{1}{p - 1} - \frac{1}{2}\frac{1}{p + 1}\right)\mathrm{d}p \quad \left[\text{By partial fractions}\right] \\ &= \frac{1}{2}\left[\ln\left(p - 1\right) - \ln\left(p + 1\right)\right]\end{aligned}$$

Putting this into the above

$$\begin{aligned}\int \frac{\sqrt{x}}{1-x^2}\mathrm{d}x &= \frac{1}{2}\Big[\ln\big(p-1\big)-\ln\big(p+1\big)\Big]+\tan^{-1}\big(p\big)\\ &= \frac{1}{2}\Big[\ln\big(\sqrt{x}-1\big)-\ln\big(\sqrt{x}+1\big)\Big]+\tan^{-1}\big(\sqrt{x}\big) \qquad \Big[\text{Because } p=\sqrt{x}\Big]\\ &= \frac{1}{2}\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\tan^{-1}\big(\sqrt{x}\big)\end{aligned}$$

Putting this $\int \frac{\sqrt{x}}{1-x^2}\mathrm{d}x = \frac{1}{2}\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\tan^{-1}\big(\sqrt{x}\big)$ into (*) gives

$$\begin{aligned}\int \frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)\mathrm{d}x &= 2\sqrt{x}\ln\left(\frac{x+1}{x-1}\right)+4\int\frac{\sqrt{x}}{1-x^2}\mathrm{d}x\\ &= 2\sqrt{x}\ln\left(\frac{x+1}{x-1}\right)+2\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+4\tan^{-1}\big(\sqrt{x}\big)\end{aligned}$$

The improper integral is given by

$$\begin{aligned}\int_2^\infty \frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)\mathrm{d}x &= \lim_{M\to\infty}\left[2\sqrt{x}\ln\left(\frac{x+1}{x-1}\right)+2\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+4\tan^{-1}\big(\sqrt{x}\big)\right]_2^M\\ &= \lim_{M\to\infty}\left[2\sqrt{M}\ln\left(\frac{M+1}{M-1}\right)+2\ln\left(\frac{\sqrt{M}-1}{\sqrt{M}+1}\right)+4\tan^{-1}\big(\sqrt{M}\big)\right]\\ &\quad -\left[2\sqrt{2}\ln\big(3\big)+2\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)+4\tan^{-1}\big(\sqrt{2}\big)\right]\end{aligned}$$

Now $\lim_{M\to\infty}\left[\ln\left(\frac{M+1}{M-1}\right)\right]=\lim_{M\to\infty}\left[\ln\left(\frac{1+\frac{1}{M}}{1-\frac{1}{M}}\right)\right]=\ln\big(1\big)=0$ and similarly

$$\lim_{M\to\infty}\left[\ln\left(\frac{\sqrt{M}-1}{\sqrt{M}+1}\right)\right]=0$$

Also $4\lim_{M\to\infty}\left[\tan^{-1}\big(\sqrt{M}\big)\right]=4\frac{\pi}{2}=2\pi$. Substituting these into the above gives

$$\int_2^\infty \frac{1}{\sqrt{x}}\ln\left(\frac{x+1}{x-1}\right)\mathrm{d}x = \big[2\pi\big]-\left[2\sqrt{2}\ln\big(3\big)+2\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)+4\tan^{-1}\big(\sqrt{2}\big)\right]$$

Hence the improper integral converges so the given series $\sum_{m=2}^{\infty}\frac{1}{\sqrt{m}}\ln\left(\frac{m+1}{m-1}\right)$ converges.