Complete Solutions to Supplementary Exercises on Infinite Series II

1. In each case we are given an alternating series and the test for an alternating series is given by:

Let $\sum_{m=1}^{\infty}(-1)^{m+1} a_m$ where a_m satisfies *all* the following:

(I) $a_m > 0$ [Positive]

(II) $\lim_{m\to\infty}(a_m) = 0$ [The mth term converges to zero as $m \to \infty$]

(III) $a_{m+1} \le a_m$ for all $m \ge N$ for some natural number N [Decreasing sequence]

Then the given alternating series $\sum_{m=1}^{\infty}(-1)^{m+1} a_m$ converges.

(a) We are asked to test $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}$ for convergence. Let $a_m = \frac{1}{2m-1}$.

Then for the natural number $m \ge 1$ we have $a_m = \frac{1}{2m-1} > 0$ (positive). Also

$$\lim_{m\to\infty}(a_m) = \lim_{m\to\infty}\left(\frac{1}{2m-1}\right) = 0$$

In order to show the given series converges we need to verify that $a_m = \frac{1}{2m-1}$ is a decreasing sequence by showing $a_{m+1} < a_m$. We have

$$2(m+1) - 1 > 2m - 1$$

$$a_{m+1} = \frac{1}{2(m+1)-1} < \frac{1}{2m-1} = a_m$$

Therefore $a_{m+1} < a_m$ for all $m \in \mathbb{N}$. By the above test we conclude that the given series $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}$ converges.

By the limit comparison test (with $\sum_{m=1}^{\infty}\frac{1}{m}$) we have that $\sum_{m=1}^{\infty}\frac{1}{2m-1}$ *diverges* so the given series $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}$ converges conditionally.

(b) This time we are given the series $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}$. Let $a_m = \frac{1}{(2m-1)^3}$.

Then clearly for $m \geq 1$ we have $a_m = \frac{1}{(2m-1)^3} > 0$. Finding the limit of this as m goes to infinity:

$$\lim_{m\to\infty}(a_m) = \lim_{m\to\infty}\left[\frac{1}{(2m-1)^3}\right] = 0$$

Check to see if we have a decreasing sequence. For $m \in \mathbb{N}$ we have

$$2(m+1)-1 > 2m-1$$
$$(2(m+1)-1)^3 > (2m-1)^3 > 0$$
$$a_{m+1} = \frac{1}{(2(m+1)-1)^3} < \frac{1}{(2m-1)^3} = a_m$$

Since $a_{m+1} < a_m$ which means that all three conditions of the alternating series test are satisfied so we conclude that $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}$ converges.

Testing for absolute convergence. Consider the series of absolute summands:

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}$$

This series converges by the p and comparison (with $\sum_{m=1}^{\infty}\frac{1}{m^3}$) tests therefore the given series is absolutely convergent.

(c) We are asked to check the convergence of $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\ln(m+1)}$. Let

$a_m = \frac{1}{\ln(m+1)}$. Then for all $m \in \mathbb{N}$ we have

$$a_m = \frac{1}{\ln(m+1)} > 0$$

Also

$$\lim_{m\to\infty}(a_m) = \lim_{m\to\infty}\left[\frac{1}{\ln(m+1)}\right] = 0$$

We also need to show that $a_m = \dfrac{1}{\ln(m+1)}$ is a decreasing sequence. We have

$a_{m+1} = \dfrac{1}{\ln(m+2)}$. We can use the derivative test to see if $\dfrac{1}{\ln}$ is decreasing.

Let $f(x) = \dfrac{1}{\ln(x)}$ then

$$f'(x) = \left(\left[\ln(x)\right]^{-1}\right)' = -\frac{1}{\ln^2(x)} \times \frac{1}{x}$$

We have $\ln^2(x) > 0$ and $x \geq 1$ which implies $f'(x) = -\dfrac{1}{x\ln^2(x)} < 0$. Hence this means that $f(x) = \dfrac{1}{\ln(x)}$ is a strictly decreasing function so

$$a_{m+1} = \frac{1}{\ln(m+2)} < \frac{1}{\ln(m+1)} = a_m \qquad [\text{because } m+2 > m+1]$$

Therefore the given series $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\ln(m+1)}$ converges.

For absolute convergence we need the absolute summand series to also converge. Consider the absolute summand series:

$$\sum_{m=1}^{\infty} \frac{1}{\ln(m+1)}$$

Now by applying the well – known inequality $\ln(x) < x$ we have

$$\ln(m+1) < m+1 \quad \Leftrightarrow \quad \frac{1}{\ln(m+1)} > \frac{1}{m+1}$$

Now by the p – test and limit comparison test $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m+1}$ diverges so by the comparison test we have $\displaystyle\sum_{m=1}^{\infty} \frac{1}{\ln(m+1)}$ diverges.

We conclude that the given series $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\ln(m+1)}$ converges conditionally.

(d) *How do we know* $\displaystyle\sum_{m=1}^{\infty} \frac{\sin\left((2m-1)\frac{\pi}{2}\right)}{m^2}$ *is an alternating series?*

Because $\sin\left(\left(2m-1\right)\frac{\pi}{2}\right)$ for $m = 1,\ 2,\ 3, \cdots$ gives

$$\sin\left(\frac{\pi}{2}\right) = 1,\ \sin\left(\frac{3\pi}{2}\right) = -1,\ \sin\left(\frac{5\pi}{2}\right) = 1,\ \sin\left(\frac{7\pi}{2}\right) = -1, \cdots$$

Let $a_m = \frac{1}{m^2}$ then for $m \ge 1$ we have

$$a_m = \frac{1}{m^2} > 0 \ \text{(Positive)}$$

Also $\lim_{m\to\infty}\left(a_m\right) = \lim_{m\to\infty}\left(\frac{1}{m^2}\right) = 0$. For convergence we need to show that a_m is a decreasing sequence for $m \ge 1$. We have

$$m+1 > m \ \Leftrightarrow\ \left(m+1\right)^2 > m^2 \ \Leftrightarrow\ a_{m+1} = \frac{1}{\left(m+1\right)^2} < \frac{1}{m^2} = a_m$$

Therefore a_m is a decreasing sequence so by the alternating series test we conclude that the given series $\sum_{m=1}^{\infty}\frac{\sin\left(\left(2m-1\right)\frac{\pi}{2}\right)}{m^2}$ converges.

For absolute convergence we need to test

$$\sum_{m=1}^{\infty}\frac{1}{m^2} \text{ which converges by the } p-\text{test.}$$

Hence the given series converges absolutely.

(e) We are given the series $\sum_{m=1}^{\infty}\frac{\left(-1\right)^{m+1}}{m\cdot 2^m}$. Let $a_m = \frac{1}{m\cdot 2^m}$ then for the natural number $m \ge 1$:

$$a_m = \frac{1}{m\cdot 2^m} > 0 \quad \text{[Positive]}$$

Moreover $\lim_{m\to\infty}\left(a_m\right) = \lim_{m\to\infty}\left(\frac{1}{m\cdot 2^m}\right) = 0$. Also

$$m < m+1 \text{ and } 2^m < 2^{m+1} \text{ implies } m\cdot 2^m < \left(m+1\right)\cdot 2^{m+1}$$

Taking the reciprocal of this gives

$$a_m = \frac{1}{m\cdot 2^m} > \frac{1}{\left(m+1\right)\cdot 2^{m+1}} = a_{m+1}$$

Or the other way round $a_{m+1} < a_m$ for all $m \geq 1$. Hence $a_m = \dfrac{1}{m \cdot 2^m}$ is a decreasing sequence.

Since all three conditions are satisfied, so by the alternating series test we conclude that $\sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{m \cdot 2^m}$ converges.

To test for absolute convergence we need to consider the series:

$$\sum_{m=1}^{\infty} \frac{1}{m \cdot 2^m}$$

We have the inequality for $x \geq 1$ that $2^x > x$. Using this we have

$$2^m > m \quad \Leftrightarrow \quad \frac{1}{2^m} < \frac{1}{m}$$

We have $\dfrac{1}{m \cdot 2^m} < \dfrac{1}{m}\dfrac{1}{m} = \dfrac{1}{m^2}$. Now the series $\sum_{m=1}^{\infty} \dfrac{1}{m^2}$ converges by the $p-$ test.

So by the comparison test $\sum_{m=1}^{\infty} \dfrac{1}{m \cdot 2^m}$ converges.

Hence $\sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{m \cdot 2^m}$ converges absolutely.

(f) We are asked to test $\sum_{m=1}^{\infty} (-1)^{m+1} \dfrac{m+1}{m}$. Let $a_m = \dfrac{m+1}{m}$ then $a_m > 0$ but

$$\lim_{m \to \infty} (a_m) = \lim_{m \to \infty} \left(\frac{m+1}{m} \right) = \lim_{m \to \infty} \left(1 + \frac{1}{m} \right) = 1 \neq 0$$

Since $\lim_{m \to \infty} (a_m) = 1 \neq 0$ so the given series $\sum_{m=1}^{\infty} (-1)^{m+1} \dfrac{m+1}{m}$ diverges because $\lim_{m \to \infty} \left[(-1)^{m+1} \dfrac{m+1}{m} \right]$ oscillates between $+1$ and -1 as m goes to infinity so the limit is *not* defined.

(g) Testing all the three conditions of the alternating series test for $\sum_{m=1}^{\infty} \dfrac{(-1)^m}{\sqrt{m}}$. Let $a_m = \dfrac{1}{\sqrt{m}}$ then $a_m > 0$ is positive and

$$\lim_{m \to \infty} (a_m) = \lim_{m \to \infty} \frac{1}{\sqrt{m}} = 0$$

For the given series to converge we also need to test that $a_m = \dfrac{1}{\sqrt{m}}$ is a decreasing sequence. We have

$$m+1 > m \quad \Leftrightarrow \quad \sqrt{m+1} > \sqrt{m} \quad \Leftrightarrow \quad a_{m+1} = \frac{1}{\sqrt{m+1}} < \frac{1}{\sqrt{m}} = a_m$$

Since $a_{m+1} < a_m$ for all $m \in \mathbb{N}$ so $a_m = \dfrac{1}{\sqrt{m}}$ is a decreasing sequence.

By the alternating series test we conclude that $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}$ converges.

Testing for absolute convergence implies we need to test $\displaystyle\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}$. This diverges by the $p-$ test. Hence the given series converges conditionally.

(h) We are asked to test $\displaystyle\sum_{m=1}^{\infty} (-1)^{m+1} \frac{m^3}{2^m}$. We use the property that if $\sum a_m$ is absolutely convergent then it is convergent. It is much easier to use this as the alternating series convergence comes automatically from the absolute convergence.

We can check it converges absolutely by applying the ratio test to $\displaystyle\sum_{m=5}^{\infty} \frac{m^3}{2^m}$. Let $a_m = \dfrac{m^3}{2^m}$ then

$$L = \lim_{m\to\infty}\left(\frac{a_{m+1}}{a_m}\right) = \lim_{m\to\infty}\left[\frac{(m+1)^3}{2^{m+1}} \times \frac{2^m}{m^3}\right]$$

$$= \frac{1}{2}\lim_{m\to\infty}\left[\left(\frac{m+1}{m}\right)^3\right] = \frac{1}{2}\lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^3\right] = \frac{1}{2} < 1$$

Since $L = \dfrac{1}{2} < 1$ so by the ratio test the given series converges absolutely.

(i) We are given $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{m - \ln(m)}$. Let $a_m = \dfrac{1}{m-\ln(m)}$. Then for all $m \in \mathbb{N}$ we have $a_m > 0$ because $m > \ln(m)$. We need to evaluate the limit:

$$\lim_{m\to\infty}(a_m) = \lim_{m\to\infty}\left(\frac{1}{m-\ln(m)}\right)$$

We have the inequalities for all $m \in \mathbb{N}$:

$$0 \le m - \ln(m) \le m \quad \Leftrightarrow \quad \frac{1}{m - \ln(m)} \ge \frac{1}{m} \qquad (*)$$

Also for all $m \in \mathbb{N}$:

$$m - \ln(m) \ge m - 2\ln(m)$$

This $m - 2\ln(m)$ is greater than or equal to zero because if $f(x) = x - 2\ln(x)$ then $f(1) = 1 - 2\ln(1) > 0$ and the derivative of this is given by

$$f'(x) = 1 - \frac{2}{x} \ge 0 \text{ for } x \ge 2$$

Since $f'(x) \ge 0$ so it is an increasing function. Hence $m - 2\ln(m) \ge 0$. We have

$$m - \ln(m) \ge m - 2\ln(m) = m - \ln(m) - \ln(m) \ge 0 \;\Rightarrow\; m - \ln(m) \ge \ln(m)$$

For $m \ge 2$ we have

$$m - \ln(m) \ge \ln(m) \quad \Leftrightarrow \quad \frac{1}{m - \ln(m)} \le \frac{1}{\ln(m)} \qquad (**)$$

Combining the two inequalities (*) and (**) gives

$$\frac{1}{m} \le \frac{1}{m - \ln(m)} \le \frac{1}{\ln(m)} \text{ for } m \ge 2$$

However $\lim_{m\to\infty}\left(\frac{1}{m}\right) = 0$ and $\lim_{m\to\infty}\left(\frac{1}{\ln(m)}\right) = 0$ $\left[\text{Because } \lim_{m\to\infty}\left[\ln(m)\right] = +\infty\right]$.

By the squeeze rule we have $\lim_{m\to\infty}(a_m) = \lim_{m\to\infty}\left(\frac{1}{m - \ln(m)}\right) = 0$.

{Squeeze Rule: If $x_m \le y_m \le z_n$ for every $m \in \mathbb{N}$ and

$$\lim_{m\to\infty}(x_m) = \lim_{m\to\infty}(z_m) = L$$

Then $\lim_{m\to\infty}(y_m) = L$.}

We also need to check that $a_m = \frac{1}{m - \ln(m)}$ is a decreasing sequence.

Let $f(x) = \frac{1}{x - \ln(x)} = \left[x - \ln(x)\right]^{-1}$ then

$$f'(x) = -\left[x - \ln(x)\right]^{-2} \times \left[1 - \frac{1}{x}\right] = -\frac{1 - \frac{1}{x}}{\left[x - \ln(x)\right]^2}$$

The derivative $f'(x)$ is negative or zero when $1-\frac{1}{x}\geq 0$. This occurs if

$$1\geq\frac{1}{x}\quad\Leftrightarrow\quad x\geq 1$$

Hence for $x\geq 1$ the derivative $f'(x)\leq 0$. This implies that $f(x)=\frac{1}{x-\ln(x)}$ is decreasing so $a_m=\frac{1}{m-\ln(m)}$ is a decreasing sequence.

By the alternating series test we conclude the given series converges.

We also need to test for absolute convergence: $\sum_{m=1}^{\infty}\frac{1}{m-\ln(m)}$

Since we have the inequality for $m\in\mathbb{N}$:

$$0\leq m-\ln(m)<m\quad\Leftrightarrow\quad\frac{1}{m-\ln(m)}>\frac{1}{m}$$

And $\sum_{m=1}^{\infty}\frac{1}{m}$ diverges because it is the harmonic series so $\sum_{m=1}^{\infty}\frac{1}{m-\ln(m)}$ diverges.

The given series $\sum_{m=1}^{\infty}\frac{(-1)^m}{m-\ln(m)}$ is conditionally convergent.

(j) We are given the series $\sum_{m=1}^{\infty}(-1)^{m+1}\frac{2m^2}{m!}$. We first check for absolute convergence because we have a factorial on the denominator so it is possible that we have absolute convergence.

We use the ratio test with $a_m=\frac{2m^2}{m!}$. Therefore $a_{m+1}=\frac{2(m+1)^2}{(m+1)!}$:

$$\begin{aligned}L&=\lim_{m\to\infty}\left(\frac{2(m+1)^2}{(m+1)!}\times\frac{m!}{2m^2}\right)\qquad\left[\text{Because } L=\lim_{m\to\infty}\left(\frac{a_{m+1}}{a_m}\right)\right]\\&=\lim_{m\to\infty}\left(\frac{\cancel{2}(m+1)^{\cancel{2}}}{\cancel{(m+1)}\cancel{m!}}\times\frac{\cancel{m!}}{\cancel{2}m^2}\right)\\&=\lim_{m\to\infty}\left(\frac{m+1}{m^2}\right)=\lim_{m\to\infty}\left(\frac{1}{m}+\frac{1}{m^2}\right)=0\end{aligned}$$

Since $L=0<1$ so by the ratio test $\sum_{m=1}^{\infty}\frac{2m^2}{m!}$ converges which implies the alternating series $\sum_{m=1}^{\infty}(-1)^{m+1}\frac{2m^2}{m!}$ converges absolutely.

2. (a) We are given the series $\sum_{m=1}^{\infty}x^m$. Let $a_m=x^m$ then $a_{m+1}=x^{m+1}$. Evaluating the limit L in the ratio test we have

$$L=\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|=\lim_{m\to\infty}\left|\frac{x^{m+1}}{x^m}\right|=\lim_{m\to\infty}|x|=|x|$$

The given series converges when

$$L=|x|<1 \iff -1<x<1$$

If $L=|x|>1 \iff x<-1$ or $x>1$ the given series diverges.

At $x=1$ we have the series $\sum_{m=1}^{\infty}1^m$ which diverges because $\lim_{m\to\infty}(1^m)=1\neq 0$.

Similarly at $x=-1$ the given series diverges.

Our interval of convergence is $-1<x<1$.

(b) We are asked to find the interval of convergence of $\sum_{m=1}^{\infty}(\ln(x))^m$.

Let $a_m=(\ln(x))^m$ then $a_{m+1}=(\ln(x))^{m+1}$ and the ratio limit is

$$L=\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|=\lim_{m\to\infty}\left|\frac{(\ln(x))^{m+1}}{(\ln(x))^m}\right|\underbrace{=}_{\text{Cancelling}}\lim_{m\to\infty}|\ln(x)|=|\ln(x)|$$

The given series converges provided

$$L=|\ln(x)|<1 \iff -1<\ln(x)<1$$

Taking exponentials gives

$$-1<\ln(x)<1 \iff e^{-1}<x<e$$

If $e^{-1}<x<e$ the given series converges. When $x=e$ our series becomes

$$\sum_{m=1}^{\infty}(\ln(x))^m=\sum_{m=1}^{\infty}(\ln(e))^m=\sum_{m=1}^{\infty}1^m \quad \left[\text{Because } \ln(e)=1\right]$$

which diverges because $\lim_{m\to\infty}(1^m)=1\neq 0$.

Similarly at $x=e^{-1}$ we have

$$\sum_{m=1}^{\infty}\left(\ln\left(x\right)\right)^{m}=\sum_{m=1}^{\infty}\left(\ln\left(e^{-1}\right)\right)^{m}=\sum_{m=1}^{\infty}\left(-1\right)^{m}$$

This diverges because $\lim_{m\to\infty}\left(-1\right)^{m}\neq 0$ as $\lim_{m\to\infty}\left(-1\right)^{m}$ is not defined.

Our interval of convergence is $e^{-1}<x<e$.

(c) We apply the ratio test to the series $\sum_{m=1}^{\infty}x^{m^2}$. Let $a_m=x^{m^2}$ then

$$a_{m+1}=x^{\left(m+1\right)^2}=x^{m^2+2m+1}$$

We have

$$L=\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|=\lim_{m\to\infty}\left|\frac{x^{m^2+2m+1}}{x^{m^2}}\right|=\lim_{m\to\infty}\left|x^{2m+1}\right|$$

The given series converges when

$$L=\lim_{m\to\infty}\left|x^{2m+1}\right|<1 \quad\Leftrightarrow\quad \left|x\right|<1 \;\Leftrightarrow\; -1<x<1$$

It is non – convergent if $x<-1,\;\; x>1$. At $x=1$ the series becomes

$$\sum_{m=1}^{\infty}1^{m^2} \quad \text{which diverges}$$

Similarly at $x=-1$ the series diverges. Our interval of convergence is $-1<x<1$.

(d) We are given the power series $\sum_{m=1}^{\infty}\frac{x^m}{m^2}$. Let $a_m=\frac{x^m}{m^2}$ then $a_{m+1}=\frac{x^{m+1}}{\left(m+1\right)^2}$:

$$L=\lim_{m\to\infty}\left|\frac{x^{m+1}}{\left(m+1\right)^2}\div\frac{x^m}{m^2}\right|$$

$$=\lim_{m\to\infty}\left|\frac{x^{\cancel{m}+1}}{\left(m+1\right)^2}\times\frac{m^2}{\cancel{x^m}}\right|=\left|x\right|\lim_{m\to\infty}\left(\frac{m}{m+1}\right)^2=\left|x\right|\left(1\right)=\left|x\right|$$

The series converges for $\left|x\right|<1 \;\Leftrightarrow\; -1<x<1$. It is non – convergent for $x>1$ or $x<-1$. At $x=1$ we have the series

$$\sum_{m=1}^{\infty}\frac{1}{m^2} \quad \text{which converges by the } p-\text{test}$$

At $x=-1$ we have $\sum_{m=1}^{\infty}\frac{\left(-1\right)^m}{m^2}$ which converges because $\sum_{m=1}^{\infty}\frac{1}{m^2}$ converges.

The interval of convergence is $-1\le x\le 1$.

(e) We are asked to find the interval of convergence of $\sum_{m=1}^{\infty} \frac{x^m}{\sqrt{m}}$. Let $a_m = \frac{x^m}{\sqrt{m}}$

then $a_{m+1} = \frac{x^{m+1}}{\sqrt{m+1}}$. The limiting ratio is given by:

$$L = \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to\infty} \left| \frac{x^{m+1}}{\sqrt{m+1}} \times \frac{\sqrt{m}}{x^m} \right|$$
$$= |x| \lim_{m\to\infty} \left(\sqrt{\frac{m}{m+1}} \right) = |x|(1) = |x|$$

By the ratio test $\sum_{m=1}^{\infty} \frac{x^m}{\sqrt{m}}$ converges for $L = |x| < 1$ which is $-1 < x < 1$ and diverges $L = |x| > 1$ and this is $x < -1,\ x > 1$.

Now we test the endpoints $x = 1$ and $x = -1$. At $x = 1$ we have the series

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \text{ which diverges}$$

At $x = -1$ we have the alternating series $\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}$. *How do we test this for convergence?*

By checking the three conditions given at the start of solution to question 1.

We have

(I) $\frac{1}{\sqrt{m}} > 0$ (Positive)

(II) $\lim_{m\to\infty} \left(\frac{1}{\sqrt{m}} \right) = 0$

(III) Decreasing sequence because for any natural number $m \geq 1$:

$$m+1 > m \iff \sqrt{m+1} > \sqrt{m} \iff \frac{1}{\sqrt{m+1}} < \frac{1}{\sqrt{m}}$$

Hence by the alternating series test $\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}$ converges.

Our interval of convergence is $-1 \leq x < 1$.

(f) We need to find the interval of convergence of $\sum_{m=1}^{\infty} \frac{1}{1+x^m}$.

For $|x| > 1$ we use the ratio test to see if it converges. Let

$$a_m = \frac{1}{1+x^m} \text{ then } a_{m+1} = \frac{1}{1+x^{m+1}}$$

Calculating the limiting ratio and seeing where it is less than 1:

$$L = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{1+x^m}{1+x^{m+1}}\right| \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{denominator by } x^{m+1}}} \lim_{m\to\infty}\left|\frac{\frac{1}{x^{m+1}}+\frac{1}{x}}{\frac{1}{x^{m+1}}+1}\right| = \lim_{m\to\infty}\left|\frac{0+\frac{1}{x}}{0+1}\right| = \frac{1}{|x|}$$

This converges provided

$$L = \frac{1}{|x|} < 1 \Leftrightarrow 1 < |x| \text{ or } |x| > 1$$

The given series $\sum_{m=1}^{\infty}\frac{1}{1+x^m}$ converges for $x < -1$ or $x > 1$.

If $|x| < 1$ the given series $\sum_{m=1}^{\infty}\frac{1}{1+x^m}$ diverges because $\lim_{m\to\infty}\left(x^m\right) = 0$ which implies $\lim_{m\to\infty}\left(\frac{1}{1+x^m}\right) = 1 \neq 0$. If $x = \pm 1$ then

$$\sum_{m=1}^{\infty}\frac{1}{1+\left(\pm 1\right)^m} \text{ diverges because } \lim_{m\to\infty}\left(\frac{1}{1+\left(\pm 1\right)^m}\right) \neq 0$$

Our set of values where the given series converges is $x < -1$ or $x > 1$.

(g) We need to find the set of values of x for which $\sum_{m=1}^{\infty} m\left(m+1\right)x^m$ converges.

We use the ratio test to see if the given series converges. Let

$$a_m = m\left(m+1\right)x^m \text{ and } a_{m+1} = \left(m+1\right)\left(m+2\right)x^{m+1}$$

The limiting ratio L is given by

$$L = \lim_{m\to\infty}\left|\frac{\left(m+1\right)\left(m+2\right)x^{m+1}}{m\left(m+1\right)x^m}\right| = |x|\lim_{m\to\infty}\left(\frac{m^2+3m+2}{m^2+m}\right)$$

$$= |x|\lim_{m\to\infty}\left(\frac{\frac{m^2}{m^2}+\frac{3m}{m^2}+\frac{2}{m^2}}{\frac{m^2}{m^2}+\frac{m}{m^2}}\right) = |x|\left(1\right) = |x|$$

By the ratio test the given series $\sum_{m=1}^{\infty} m\left(m+1\right)x^m$ converges for $L = |x| < 1$ and diverges $L = |x| > 1$.

We also have non – convergence of the given series $\sum_{m=1}^{\infty} m\left(m+1\right)x^m$ when $x = \pm 1$ because at $x = 1$:

$$\lim_{m\to\infty}\left(m\left(m+1\right)1^m\right) = +\infty \neq 0$$

And at $x = -1$ the mth term

$$\lim_{m\to\infty}\left(m\left(m+1\right)\left(-1\right)^m\right) \text{ is not defined so cannot equal zero.}$$

Our interval of convergence is $-1 < x < 1$.

(h) We are asked to find the set of values of x such that $\sum_{m=1}^{\infty} \frac{x^m}{m+\sqrt{m}}$ converges.

How do we find these x values?

By the ratio test with $a_m = \frac{x^m}{m+\sqrt{m}}$ and $a_{m+1} = \frac{x^{m+1}}{\left(m+1\right)+\sqrt{m+1}}$. We have

$$L = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{x^{m+1}}{\left(m+1\right)+\sqrt{m+1}} \times \left(\frac{m+\sqrt{m}}{x^m}\right)\right|$$

$$= \left|x\right| \lim_{m\to\infty}\left|\frac{m+\sqrt{m}}{\left(m+1\right)+\sqrt{m+1}}\right| = \left|x\right|\left(1\right) = \left|x\right|$$

The given series $\sum_{m=1}^{\infty} \frac{x^m}{m+\sqrt{m}}$ converges for $\left|x\right| < 1$ or $-1 < x < 1$ and diverges when $\left|x\right| > 1$. At $x = 1$ we have the series $\sum_{m=1}^{\infty} \frac{1}{m+\sqrt{m}}$. We need to test this for convergence. *How?*

We use the limit comparison test with $b_m = \frac{1}{m}$ and $a_m{}^* = \frac{1}{m+\sqrt{m}}$ so

$$\lim_{m\to\infty}\left(\frac{a_m{}^*}{b_m}\right) = \lim_{m\to\infty}\left(\frac{m}{m+\sqrt{m}}\right) = 1$$

Since the harmonic series diverges so $\sum_{m=1}^{\infty} \frac{1}{m+\sqrt{m}}$ diverges.

At $x = -1$ we have the *alternating* series $\sum_{m=1}^{\infty} \frac{\left(-1\right)^m}{m+\sqrt{m}}$. Using the alternating series test with $a_m{}^* = \frac{1}{m+\sqrt{m}}$ we have

(I) $a_m{}^* = \dfrac{1}{m+\sqrt{m}} > 0$ for all natural numbers m

(II) $\lim_{m\to\infty}\left(a_m{}^*\right) = \lim_{m\to\infty}\left(\dfrac{1}{m+\sqrt{m}}\right) = 0$

(III) Since

$$m+1+\sqrt{m+1} > m+\sqrt{m} \quad \Leftrightarrow \quad \frac{1}{m+1+\sqrt{m+1}} < \frac{1}{m+\sqrt{m}}$$

So we have a decreasing sequence.

Because *all* three conditions are satisfied, so by the alternating series test

$\sum_{m=1}^{\infty}\dfrac{(-1)^m}{m+\sqrt{m}}$ converges.

Our interval of convergence is $-1 \le x < 1$.

(i) We are asked to test the values of x for which $\sum_{m=1}^{\infty}\dfrac{x^m}{1+x^{2m}}$ converges.

Using the ratio test with $a_m = \dfrac{x^m}{1+x^{2m}}$ and $a_{m+1} = \dfrac{x^{m+1}}{1+x^{2m+2}}$ we have

$$L = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{x^{m+1}}{1+x^{2m+2}} \div \frac{x^m}{1+x^{2m}}\right|$$

$$= \lim_{m\to\infty}\left|\frac{x^{m+1}}{1+x^{2m+2}} \times \frac{1+x^{2m}}{x^m}\right| = |x|\lim_{m\to\infty}\left|\frac{1+x^{2m}}{1+x^{2m+2}}\right| \qquad (*)$$

If $|x| < 1$ then $L = |x|\lim_{m\to\infty}\left|\dfrac{1+x^{2m}}{1+x^{2m+2}}\right| < 1$ because $\lim_{m\to\infty}\left(x^{2m}\right) = \lim_{m\to\infty}\left(x^{2m+2}\right) = 0$

and so $\sum_{m=1}^{\infty}\dfrac{x^m}{1+x^{2m}}$ converges. Hence we have convergence for $-1 < x < 1$.

If $|x| > 1$ then we evaluate the last term on the right hand side of (*) by dividing the numerator and denominator by the dominant term x^{2m+2}:

$$\lim_{m\to\infty}\left|\frac{1+x^{2m}}{1+x^{2m+2}}\right| = \lim_{m\to\infty}\left|\frac{\frac{1}{x^{2m+2}}+\frac{x^{2m}}{x^{2m+2}}}{\frac{1}{x^{2m+2}}+1}\right| = \lim_{m\to\infty}\left|\frac{0+\frac{1}{x^2}}{0+1}\right| = \frac{1}{|x^2|}$$

Substituting this $\lim_{m\to\infty}\left|\dfrac{1+x^{2m}}{1+x^{2m+2}}\right| = \dfrac{1}{|x^2|}$ into (*) gives

$$L = \left|x\right|\frac{1}{\left|x^2\right|} = \frac{1}{\left|x\right|} < 1 \quad \Leftrightarrow \quad 1 < \left|x\right| \quad \text{or} \quad \left|x\right| > 1 \quad \Leftrightarrow \quad x < -1, \ \ x > 1$$

If $x = 1$ then we have the series $\sum_{m=1}^{\infty} \frac{1^m}{1+1^{2m}}$ which diverges because as $m \to \infty$

so $\frac{1^m}{1+1^{2m}} \to \frac{1}{2} \neq 0$.

If $x = -1$ then we have the series $\sum_{m=1}^{\infty} \frac{\left(-1\right)^m}{2}$ which diverges because as $m \to \infty$

so $\frac{\left(-1\right)^m}{2}$ is *not* defined.

The set of values of x for which the given series $\sum_{m=1}^{\infty} \frac{x^m}{1+x^{2m}}$ converges is

$$-1 < x < 1 \quad \text{or} \quad x > 1 \quad \text{or} \quad x < -1$$

(j) We are given the power series $\sum_{m=1}^{\infty} \sin\left(\frac{x}{2^m}\right)$. Using the ratio test with

$a_m = \sin\left(\frac{x}{2^m}\right)$ and so $a_{m+1} = \sin\left(\frac{x}{2^{m+1}}\right)$:

$$L = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{\sin\left(\frac{x}{2^{m+1}}\right)}{\sin\left(\frac{x}{2^m}\right)}\right| = \lim_{m\to\infty}\left|\frac{\sin\left(\frac{1}{2}\frac{x}{2^m}\right)}{\sin\left(\frac{x}{2^m}\right)}\right| \qquad (*)$$

Using the double angle formula:

(4.53) $$\sin\left(2A\right) = 2\sin\left(A\right)\cos\left(A\right)$$

Applying this to $\sin\left(\frac{x}{2^m}\right)$ gives

$$\sin\left(\frac{x}{2^m}\right) = 2\sin\left(\frac{1}{2}\frac{x}{2^m}\right)\cos\left(\frac{1}{2}\frac{x}{2^m}\right)$$

Substituting this $\sin\left(\frac{x}{2^m}\right) = 2\sin\left(\frac{1}{2}\frac{x}{2^m}\right)\cos\left(\frac{1}{2}\frac{x}{2^m}\right)$ into the above (*) yields

$$L = \lim_{m\to\infty}\left|\frac{\cancel{\sin\left(\frac{1}{2}\frac{x}{2^m}\right)}}{2\cancel{\sin\left(\frac{1}{2}\frac{x}{2^m}\right)}\cos\left(\frac{1}{2}\frac{x}{2^m}\right)}\right| = \frac{1}{2}\lim_{m\to\infty}\frac{1}{\left|\cos\left(\frac{1}{2}\frac{x}{2^m}\right)\right|} = \frac{1}{2} \qquad \left[\text{Because } \lim_{m\to\infty}\left|\cos\left(\frac{1}{2}\frac{x}{2^m}\right)\right| = \cos\left(0\right) = 1\right]$$

Hence the given series $\sum_{m=1}^{\infty} \sin\left(\frac{x}{2^m}\right)$ converges for all real x, because $L = \frac{1}{2} < 1$, that is x satisfies $-\infty < x < +\infty$.

(k) We are asked to find the set of values of x for which $\sum_{m=1}^{\infty} x^m \tan\left(\frac{x}{2^m}\right)$ converges. Let $a_m = x^m \tan\left(\frac{x}{2^m}\right)$ then $a_{m+1} = x^{m+1} \tan\left(\frac{x}{2^{m+1}}\right)$:

$$L = \lim_{m\to\infty} \left| \frac{x^{m+1} \tan\left(\frac{x}{2^{m+1}}\right)}{x^m \tan\left(\frac{x}{2^m}\right)} \right| = |x| \lim_{m\to\infty} \left| \frac{\tan\left(\frac{1}{2}\frac{x}{2^m}\right)}{\tan\left(\frac{x}{2^m}\right)} \right| \qquad (\dagger)$$

The double angle formula for tan is given by:

$$(4.55) \qquad \tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

Applying this to $\tan\left(\frac{x}{2^m}\right)$ gives

$$\tan\left(\frac{x}{2^m}\right) = \frac{2\tan\left(\frac{1}{2}\frac{x}{2^m}\right)}{1-\tan^2\left(\frac{1}{2}\frac{x}{2^m}\right)} = \frac{2t}{1-t^2} \quad \text{where} \quad t = \tan\left(\frac{1}{2}\frac{x}{2^m}\right)$$

Substituting this into (†) gives

$$L = |x| \lim_{m\to\infty} \left| \frac{\not{t}}{2\not{t} \,/\, \left(1-t^2\right)} \right| = \frac{|x|}{2} \lim_{m\to\infty} \left| 1 - \tan^2\left(\frac{1}{2}\frac{x}{2^m}\right) \right| = \frac{|x|}{2}(1-0) = \frac{|x|}{2}$$

By the ratio test $\sum_{m=1}^{\infty} x^m \tan\left(\frac{x}{2^m}\right)$ converges for $L = \frac{|x|}{2} < 1$ which implies

$$\frac{|x|}{2} < 1 \;\Leftrightarrow\; |x| < 2 \;\Leftrightarrow\; -2 < x < 2$$

Additionally $\sum_{m=1}^{\infty} x^m \tan\left(\frac{x}{2^m}\right)$ diverges for $x > 2$ or $x < -2$.

At $x = 2$ we have the series $\sum_{m=1}^{\infty} 2^m \tan\left(\frac{2}{2^m}\right) = \sum_{m=1}^{\infty} 2^m \tan\left(\frac{1}{2^{m-1}}\right)$. The limiting value of $2^m \tan\left(\frac{1}{2^{m-1}}\right)$ as m goes to infinity is denoted by $\lim_{m\to\infty}\left(2^m \tan\left(\frac{1}{2^{m-1}}\right)\right)$. If this limiting value is non-zero then we can conclude the series does not converge at this point $x=2$. Using the power series for *tan* which is

(7.18) $$\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots \qquad -\frac{\pi}{2}<x<\frac{\pi}{2}$$

Substituting $x=\dfrac{1}{2^{m-1}}$ into this gives

$$\tan\left(\frac{1}{2^{m-1}}\right)=\frac{1}{2^{m-1}}+\frac{1}{3}\left(\frac{1}{2^{m-1}}\right)^3+\frac{2}{15}\left(\frac{1}{2^{m-1}}\right)^5+\frac{17}{315}\left(\frac{1}{2^{m-1}}\right)^7+\cdots$$

Therefore

$$\begin{aligned}\lim_{m\to\infty}\left(2^m\tan\left(\frac{1}{2^{m-1}}\right)\right)&=\lim_{m\to\infty}\left(2^m\left(\frac{1}{2^{m-1}}+\frac{1}{3}\left(\frac{1}{2^{m-1}}\right)^3+\frac{2}{15}\left(\frac{1}{2^{m-1}}\right)^5+\frac{17}{315}\left(\frac{1}{2^{m-1}}\right)^7+\cdots\right)\right)\\&=\lim_{m\to\infty}\left(2+\frac{1}{3}\left(\frac{2^m}{2^{3m-3}}\right)+\frac{2}{15}\left(\frac{2^m}{2^{5m-5}}\right)+\frac{17}{315}\left(\frac{2^m}{2^{7m-7}}\right)+\cdots\right)\\&\underbrace{=}_{\text{Applying the rules of indices}}\lim_{m\to\infty}\left(2+\frac{1}{3}\left(\frac{1}{2^{2m-3}}\right)+\frac{2}{15}\left(\frac{1}{2^{4m-5}}\right)+\frac{17}{315}\left(\frac{1}{2^{6m-7}}\right)+\cdots\right)\\&=2+0+0+0+\cdots=2\end{aligned}$$

Since the limiting term does not go to zero, so at $x=2$ the given series $\sum_{m=1}^{\infty}x^m\tan\left(\frac{x}{2^m}\right)$ diverges.

At $x=-2$ we have the series

$$\sum_{m=1}^{\infty}(-2)^m\tan\left(\frac{-2}{2^m}\right)=\sum_{m=1}^{\infty}(-2)^m\tan\left(-\frac{1}{2^{m-1}}\right)$$

Using the property

(4.52) $$\tan(-x)=-\tan(x)$$

We have

$$\begin{aligned}\sum_{m=1}^{\infty}(-2)^m\tan\left(\frac{-2}{2^m}\right)&=\sum_{m=1}^{\infty}(-2)^m(-1)\tan\left(\frac{1}{2^{m-1}}\right)\qquad [\text{By } (4.52)]\\&=\sum_{m=1}^{\infty}(-1)^m(-1)2^m\tan\left(\frac{1}{2^{m-1}}\right)=\sum_{m=1}^{\infty}(-1)^{m+1}2^m\tan\left(\frac{1}{2^{m-1}}\right)\end{aligned}$$

This is an alternating series.

Since $2^m\tan\left(\frac{1}{2^{m-1}}\right)$ is equal to 2 as $m\to\infty$ so $\lim_{m\to\infty}\left[(-1)^{m+1}2^m\tan\left(\frac{1}{2^{m-1}}\right)\right]$ does *not* exist so the given series diverges.

Hence the given series $\sum_{m=1}^{\infty} x^m \tan\left(\frac{x}{2^m}\right)$ converges for $-2 < x < 2$.

3. To find the interval of convergence we use the ratio test.

(a) We are given $\sum_{m=1}^{\infty} 10^m x^m$. Let $a_m = 10^m x^m$ then $a_{m+1} = 10^{m+1} x^{m+1}$ and

$$L = \lim_{m\to\infty} \left|\frac{10^{m+1} x^{m+1}}{10^m\ x^m}\right| = 10|x|$$

The series converges if this limit is less than 1, that is

$$L = 10|x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{10}$$

The given series $\sum_{m=1}^{\infty} 10^m x^m$ converges if $-\frac{1}{10} < x < \frac{1}{10}$ and diverges if $x > \frac{1}{10}$ or $x < -\frac{1}{10}$.

We now check the endpoints. At $x = \frac{1}{10}$ we have the series

$$\sum_{m=1}^{\infty} 10^m \left(\frac{1}{10}\right)^m = \sum_{m=1}^{\infty} 1^m \text{ which diverges because } \lim_{m\to\infty}\left(1^m\right) = 1 \neq 0$$

Similarly at $x = -\frac{1}{10}$ we have

$$\sum_{m=1}^{\infty} 10^m \left(-\frac{1}{10}\right)^m = \sum_{m=1}^{\infty} (-1)^m \text{ which diverges because } \lim_{m\to\infty}\left(-1^m\right) \text{ is not defined}$$

Our interval of convergence is $-\frac{1}{10} < x < \frac{1}{10}$.

(b) We need to find the interval of convergence of $\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m$. Let $a_m = \frac{(-1)^{m+1}}{m} x^m$ then $a_{m+1} = \frac{(-1)^{m+2}}{m+1} x^{m+1}$. Therefore

$$L = \lim_{m\to\infty} \left|\frac{(-1)^{m+2}\ x^{m+1}}{m+1} \div \frac{(-1)^{m+1}\ x^m}{m}\right|$$

$$= \lim_{m\to\infty} \left|\frac{(-1)^{m+2}\ x^{m+1}}{m+1} \times \frac{m}{(-1)^{m+1}\ x^m}\right| = |x| \lim_{m\to\infty}\left(\frac{m}{m+1}\right) = |x|(1) = |x|$$

Hence the given series converges absolutely for $\left|x\right| < 1$ or $-1 < x < 1$. For $x > 1$ or $x < -1$ the series diverges.

At the endpoint $x = 1$ we have the series

$$\sum_{m=1}^{\infty} \frac{\left(-1\right)^{m+1}}{m}\left(1\right)^{m} = \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m+1}}{m}$$

This series converges because it is the alternating harmonic series.

At $x = -1$ we have the series

$$\sum_{m=1}^{\infty} \frac{\left(-1\right)^{m+1}}{m}\left(-1\right)^{m} = \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m+1+m}}{m} = \sum_{m=1}^{\infty} \frac{\left(-1\right)^{2m+1}}{m} \underbrace{=}_{\text{because the index is odd}} \sum_{m=1}^{\infty}\left(-\frac{1}{m}\right)$$

This diverges because it is harmonic series with a minus sign.

Our interval of convergence is $-1 < x \le 1$.

(c) We need to find the interval of convergence of $\sum_{m=0}^{\infty} m!x^{m}$. Let $a_{m} = m!x^{m}$ then $a_{m+1} = \left(m+1\right)!x^{m+1}$ and limiting ratio is given by

$$L = \lim_{m\to\infty}\left|\frac{\left(m+1\right)!x^{m+1}}{m!x^{m}}\right| = \left|x\right|\lim_{m\to\infty}\left|\frac{\left(m+1\right)m!}{m!}\right| = +\infty$$

Therefore the given series converges only at the point $x = 0$.

(d) We are asked to find the interval of convergence of $\sum_{m=0}^{\infty} 2^{m}x^{2m}$. Let $a_{m} = 2^{m}x^{2m}$ then $a_{m+1} = 2^{m+1}x^{2m+2}$. Applying the ratio test gives

$$L = \lim_{m\to\infty}\left|\frac{2^{m+1}x^{2m+2}}{2^{m}x^{2m}}\right| = 2x^{2}$$

The series converges if this is less than 1, so it converges when x satisfies

$$L = 2x^{2} < 1 \;\Leftrightarrow\; x^{2} < \frac{1}{2} \;\Leftrightarrow\; \left|x\right| < \frac{1}{\sqrt{2}} \;\Leftrightarrow\; -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

At the endpoint $x = \frac{1}{\sqrt{2}}$ we have

$$\sum_{m=0}^{\infty} 2^{m}\left(\frac{1}{\sqrt{2}}\right)^{2m} = \sum_{m=0}^{\infty} 2^{m}\left(\frac{1}{2}\right)^{m} = \sum_{m=0}^{\infty} 1^{m} \text{ which diverges}$$

At the other endpoint $x = -\frac{1}{\sqrt{2}}$ we have

$$\sum_{m=0}^{\infty} 2^m \left(-\frac{1}{\sqrt{2}}\right)^{2m} = \sum_{m=0}^{\infty} 2^m \left(\frac{1}{2}\right)^m = \sum_{m=0}^{\infty} 1^m \text{ diverges}$$

Our interval of convergence is $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

(e) We are asked to determine the interval of convergence of

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)(2m-1)!} x^{2m-1}$$

We consider the absolute series

$$\sum_{m=1}^{\infty} \left| \frac{(-1)^{m+1}}{(2m-1)(2m-1)!} x^{2m-1} \right| = \sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1)(2m-1)!}$$

Let $a_m = \frac{x^{2m-1}}{(2m-1)(2m-1)!}$ then $a_{m+1} = \frac{x^{2m+1}}{(2m+1)(2m+1)!}$. Substituting these into the ratio test formula gives

$$\begin{aligned}
L &= \lim_{m\to\infty} \left| \frac{x^{2m+1}}{(2m+1)(2m+1)!} \times \frac{(2m-1)(2m-1)!}{x^{2m-1}} \right| \\
&= \lim_{m\to\infty} \left| \frac{x^{\cancel{2m-1}+2}}{(2m+1)(2m+1)(2m)\cancel{(2m-1)!}} \times \frac{(2m-1)\cancel{(2m-1)!}}{\cancel{x^{2m-1}}} \right| \\
&= x^2 \lim_{m\to\infty} \left(\frac{2m-1}{8m^2+8m+2m} \right) \\
&= x^2 \lim_{m\to\infty} \left(\frac{\frac{2}{m} - \frac{1}{m^2}}{8 + \frac{8}{m} + \frac{2}{m}} \right) = x^2(0) = 0 \quad \left[\begin{array}{l}\text{Dividing numerator and} \\ \text{denominator by } m^2\end{array}\right]
\end{aligned}$$

The given series converges for all x because $L = 0 < 1$ and so the interval of convergence is $-\infty < x < +\infty$.

(f) We are given $\sum_{m=1}^{\infty} m^m x^m$. Let $a_m = m^m x^m$ then $a_{m+1} = (m+1)^{m+1} x^{m+1}$ and

$$\begin{aligned}
L &= \lim_{m\to\infty} \left| \frac{(m+1)^{m+1} x^{\cancel{m}+1}}{m^m \cancel{x^m}} \right| \\
&= |x| \lim_{m\to\infty} \left[(m+1) \left(\frac{m+1}{m} \right)^m \right] = |x| \lim_{m\to\infty} \left[(m+1) \left(1 + \frac{1}{m} \right)^m \right]
\end{aligned}$$

We have the well – known limit $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$. Therefore

$$L = |x|\lim_{m\to\infty}\left[(m+1)\left(1+\frac{1}{m}\right)^m\right] = +\infty$$

The series converges only at $x = 0$.

(g) We are given the series $\sum_{m=1}^{\infty}\frac{\ln(m+1)}{m+1}x^{m+1}$. For interval of convergence we use the ratio test with

$$a_m = \frac{\ln(m+1)}{m+1}x^{m+1} \text{ which implies } a_{m+1} = \frac{\ln(m+2)}{m+2}x^{m+2}$$

The limiting ratio is given by

$$L = \lim_{m\to\infty}\left|\frac{\ln(m+2)x^{m+2}}{m+2}\times\frac{m+1}{\ln(m+1)x^{m+1}}\right|$$

$$= |x|\lim_{m\to\infty}\left[\left(\frac{m+1}{m+2}\right)\left(\frac{\ln(m+2)}{\ln(m+1)}\right)\right] \qquad (*)$$

Considering each of the limits separately we have

$$\lim_{m\to\infty}\left(\frac{m+1}{m+2}\right) = \lim_{m\to\infty}\left(\frac{1+\frac{1}{m}}{1+\frac{2}{m}}\right) = 1 \qquad \left[\text{Dividing numerator and denominator by } m\right]$$

The other limit we find by L'hopital's rule:

$$\lim_{m\to\infty}\left(\frac{\ln(m+2)}{\ln(m+1)}\right) = \lim_{m\to\infty}\left(\frac{1/(m+2)}{1/(m+1)}\right) = \lim_{m\to\infty}\left(\frac{m+1}{m+2}\right) = 1$$

Substituting these two limits $\lim_{m\to\infty}\left(\frac{m+1}{m+2}\right) = 1$ and $\lim_{m\to\infty}\left(\frac{\ln(m+2)}{\ln(m+1)}\right) = 1$ into (*) gives

$$L = |x|\lim_{m\to\infty}\left[\left(\frac{m+1}{m+2}\right)\left(\frac{\ln(m+2)}{\ln(m+1)}\right)\right] = |x|$$

The given series converges if $L = |x| < 1$ or $-1 < x < 1$.

At the endpoint $x = 1$ we have the series

$$\sum_{m=1}^{\infty} \frac{\ln(m+1)}{m+1}$$

We need to test this for convergence. We can rewrite this series as

$$\sum_{m=1}^{\infty} \frac{\ln(m+1)}{m+1} = \frac{\ln(2)}{2} + \sum_{m=2}^{\infty} \frac{\ln(m+1)}{m+1}$$

For $m \geq 2$ we have the inequality $\ln(m+1) > 1$ and

$$\frac{\ln(m+1)}{m+1} > \frac{1}{m+1}$$

Since the series $\sum_{m=2}^{\infty} \frac{1}{m+1}$ diverges so by the comparison test $\sum_{m=2}^{\infty} \frac{\ln(m+1)}{m+1}$ diverges. Hence at $x = 1$ the given series diverges.

At the other endpoint $x = -1$ we have the series

$$\sum_{m=1}^{\infty} \frac{\ln(m+1)}{m+1}(-1)^{m+1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\ln(m+1)}{m+1}$$

This is an alternating series. Using the alternating series test with

$$b_m = \frac{\ln(m+1)}{m+1}$$

For $m \geq 1$ we have $b_m = \frac{\ln(m+1)}{m+1} > 0$ and

$$\lim_{m\to\infty} (b_m) = \lim_{m\to\infty} \left(\frac{\ln(m+1)}{m+1} \right) = \lim_{m\to\infty} \left(\frac{1}{m+1} \right) = 0 \quad \left[\text{By L'hopital's rule}\right]$$

For decreasing sequence we let $f(x) = \frac{\ln(x+1)}{x+1}$ then

$$f'(x) = \frac{\frac{1}{x+1}(x+1) - \ln(x+1)}{(x+1)^2} = \frac{1 - \ln(x+1)}{(x+1)^2}$$

For $x \geq 2$ we have $f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0$. This implies that the sequence

$b_m = \frac{\ln(m+1)}{m+1}$ is decreasing from $m \geq 2$.

By the alternating series test we conclude that at the endpoint $x = -1$ the given series converges.

The interval of convergence is $-1 \le x < 1$.

4. (i) We need to find the interval of convergence of $\tan^{-1}(x) = \sum_{m=0}^{\infty}(-1)^m \frac{x^{2m+1}}{2m+1}$.

Consider the absolute series $\sum_{m=0}^{\infty}\left|(-1)^m \frac{x^{2m+1}}{2m+1}\right| = \sum_{m=0}^{\infty}\frac{x^{2m+1}}{2m+1}$.

Let $a_m = \frac{x^{2m+1}}{2m+1}$ then $a_{m+1} = \frac{x^{2m+3}}{2m+3}$ and so

$$L = \lim_{m\to\infty}\left|\frac{x^{2m+3}}{2m+3} \times \frac{2m+1}{x^{2m+1}}\right| = \left|x^2\right| \lim_{m\to\infty}\left(\frac{2m+1}{2m+3}\right) = x^2$$

This series converges absolutely if $L = x^2 < 1 \iff -1 < x < 1$.

At the endpoints $x = \pm 1$ we have the series

$$\tan^{-1}(\pm 1) = \sum_{m=0}^{\infty}(-1)^m \frac{(\pm 1)^{2m+1}}{2m+1}$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$ so the given series converges.

Our interval of convergence is $-1 \le x \le 1$.

(ii) From (7.21) we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad -1 < x \le 1$$

Replacing the x with $-x$ gives

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \quad 1 > x \ge -1 \text{ or } -1 \le x < 1$$

Subtracting the last two series gives

$$\begin{aligned}\ln(1+x) - \ln(1-x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)\\ &= \underbrace{x + x}_{=2x} \underbrace{- \frac{x^2}{2} + \frac{x^2}{2}}_{=0} + \underbrace{\frac{x^3}{3} + \frac{x^3}{3}}_{=2\frac{x^3}{3}} \underbrace{- \frac{x^4}{4} + \frac{x^4}{4}}_{=0} + \underbrace{\frac{x^5}{5} + \frac{x^5}{5}}_{=2\frac{x^5}{5}} + \cdots\\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)\end{aligned}$$

This series converges provided that $-1 < x < 1$.

Dividing both sides by 2 and using the laws of logs; $\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$ we have

$$\frac{1}{2}\frac{\ln(1+x)}{\ln(1-x)} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

(iii) We are asked to find the sum of the series $\sum_{m=1}^{\infty}\frac{x^{4m-3}}{4m-3}$. Writing this out:

$$\sum_{m=1}^{\infty}\frac{x^{4m-3}}{4m-3} = x + \frac{x^5}{5} + \frac{x^9}{9} + \frac{x^{13}}{13} + \cdots$$

We are given $\tan^{-1}(x) = \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{2m+1}$ which in expanded form is

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (*)$$

Adding the series we found in part (ii) to (*) gives

$$\tan^{-1}(x) + \frac{1}{2}\frac{\ln(1+x)}{\ln(1-x)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$
$$= 2\left(x + \frac{x^5}{5} + \frac{x^9}{9} + \cdots\right)$$

Taking half of this gives us the required series

$$\frac{1}{2}\left[\tan^{-1}(x) + \frac{1}{2}\frac{\ln(1+x)}{\ln(1-x)}\right] = x + \frac{x^5}{5} + \frac{x^9}{9} + \cdots = \sum_{m=1}^{\infty}\frac{x^{4m-3}}{4m-3}$$

The interval of convergence is $-1 < x < 1$ because both $\frac{\ln(1+x)}{\ln(1-x)}$ and $\tan^{-1}(x)$ have interval of convergences of $-1 < x < 1$ and $-1 \le x \le 1$ respectively.

5. We are asked to find the sum of $\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m(m+1)}$. Expanding this out gives

$$\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m(m+1)} = \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{12} - \frac{x^4}{20} + \frac{x^5}{30} - \cdots$$

Using the partial fractions:

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

We have

$$\begin{aligned}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m(m+1)} &= \sum_{m=1}^{\infty}(-1)^{m+1}x^m\left[\frac{1}{m}-\frac{1}{m+1}\right]\\ &= x\underbrace{\left[1-\frac{1}{2}\right]}_{m=1}-x^2\underbrace{\left[\frac{1}{2}-\frac{1}{3}\right]}_{m=2}+x^3\underbrace{\left[\frac{1}{3}-\frac{1}{4}\right]}_{m=3}-x^4\underbrace{\left[\frac{1}{4}-\frac{1}{5}\right]}_{m=4}+x^5\underbrace{\left[\frac{1}{5}-\frac{1}{6}\right]}_{m=5}+\cdots\\ &= \underbrace{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots}_{=\ln(1+x)}-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\cdots \qquad (*)\end{aligned}$$

If we expand $\dfrac{\ln(1+x)}{x}$ by using the power series of the natural logarithm:

$$\begin{aligned}\frac{\ln(1+x)}{x} &= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots\right)\\ &= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\cdots\end{aligned}$$

These terms apart from 1 are the same terms on the right hand side of (*).

Taking away 1 gives us our result:

$$\begin{aligned}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m(m+1)} &= \ln(1+x)\underbrace{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\cdots}_{=\frac{\ln(1+x)}{x}-1}\\ &= \ln(1+x)+\frac{\ln(1+x)}{x}-1\\ &= \frac{x\ln(1+x)+\ln(1+x)-x}{x}\\ &= \frac{(x+1)\ln(1+x)-x}{x}\end{aligned}$$

The interval of convergence is $-1<x\le 1$ and $x\ne 0$ because that is the interval of convergence of $\ln(1+x)$.