Complete Solutions to Supplementary Exercises on Indefinite Integration

1. (a) We are given the indefinite integral $\int \frac{x}{x+4}\,\mathrm{d}x$. Applying long division to the integrand:

$$\begin{array}{r|l} & 1 \\ x+4 & x \\ & -(x+4) \\ \hline & 0-4 \end{array}$$

We have $\frac{x}{x+4} = 1 - \frac{4}{x+4}$. Therefore

$$\int \frac{x}{x+4}\,\mathrm{d}x = \int\left(1-\frac{4}{x+4}\right)\mathrm{d}x = x - 4\ln\left|x+4\right| + C$$

(b) We are asked to find the integral $\int \frac{x}{2x+1}\,\mathrm{d}x$. Using long division

$$\begin{array}{r|l} & \frac{1}{2} \\ 2x+1 & x \\ & -\left(x+\frac{1}{2}\right) \\ \hline & 0-\frac{1}{2} \end{array}$$

The integrand can be expressed as

$$\frac{x}{2x+1} = \frac{1}{2} - \frac{1/2}{2x+1} = \frac{1}{2}\left[1-\frac{1}{2x+1}\right]$$

Integrating

$$\int \frac{x}{2x+1}\,\mathrm{d}x = \frac{1}{2}\int\left[1-\frac{1}{2x+1}\right]\mathrm{d}x = \frac{1}{2}\left[x-\frac{1}{2}\ln\left|2x+1\right|\right]+C = \frac{1}{4}\left[2x-\ln\left|2x+1\right|\right]+C$$

(c) We are asked to find $\int \frac{x^2-1}{x^2+1}\,\mathrm{d}x$. By long division we have

$$\begin{array}{r|l} & 1 \\ x^2+1 & x^2-1 \\ & -\left(x^2+1\right) \\ \hline & 0\ \ -2 \end{array}$$

We can express the integrand as

$$\frac{x^2-1}{x^2+1}=1-\frac{2}{x^2+1}$$

Integrating the right hand side gives

$$\int\frac{x^2-1}{x^2+1}\,\mathrm{d}x=\int\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x=x-2\underbrace{\tan^{-1}\left(x\right)}_{\text{by (8.26)}}+C$$

(d) We are given $\int\frac{\left(1+x\right)^2}{x^2+1}\,\mathrm{d}x$. Expanding the numerator of the integrand gives

$$\int\frac{\left(1+x\right)^2}{x^2+1}\,\mathrm{d}x=\int\frac{x^2+2x+1}{x^2+1}\,\mathrm{d}x=\int\frac{\cancel{x^2+1}}{\cancel{x^2+1}}\,\mathrm{d}x+\int\frac{2x}{x^2+1}\,\mathrm{d}x\,.$$

Alternatively by long division we have

$$\begin{array}{r|l} & \quad\quad\quad\quad\;\; 1 \\ x^2+1 & \;x^2+2x+1 \\ & -\left(x^2\quad\quad+1\right) \\ \hline & \quad\quad 2x \end{array}$$

Hence $\frac{x^2+2x+1}{x^2+1}=1+\frac{2x}{x^2+1}$. Integrating this gives

$$\int\frac{\left(1+x\right)^2}{x^2+1}\,\mathrm{d}x=\int\left(1+\frac{2x}{x^2+1}\right)\mathrm{d}x=x+\ln\left|x^2+1\right|+C$$

(e) We need to find $\int\frac{Ax}{a+bx}\,\mathrm{d}x$. By long division we have

$$\begin{array}{r|l} & \quad\quad\quad\quad A/b \\ a+bx & \;Ax \\ & -\left(Ax+\frac{aA}{b}\right) \\ \hline & \;0\quad -\frac{aA}{b} \end{array}$$

Expressing the integrand as

$$\frac{Ax}{a+bx}=\frac{A}{b}-\frac{aA}{b\left(a+bx\right)}=\frac{A}{b}\left[1-\frac{a}{a+bx}\right]$$

Integrating this yields

$$\int \frac{Ax}{a+bx}\,\mathrm{d}x = \frac{A}{b}\int\left[1-\frac{a}{a+bx}\right]\mathrm{d}x = \frac{A}{b}\left[x-\frac{a}{b}\ln\left|a+bx\right|\right]+C$$
$$= \frac{A}{b^2}\left[bx - a\ln\left|a+bx\right|\right]+C$$

(f) We need to find $\int \frac{x^4}{x^2+1}\,\mathrm{d}x$. Applying long division to the integrand:

$$\begin{array}{r|l} & x^2-1 \\ \hline x^2+1 & x^4 \\ & -\left(x^4+x^2\right) \\ \hline & 0 \quad -x^2 \\ & -\left(-x^2-1\right) \\ \hline & 1 \end{array}$$

Expressing the given integrand using this yields

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

Now integrating the right hand side gives

$$\int \frac{x^4}{x^2+1}\,\mathrm{d}x = \int\left(x^2-1+\frac{1}{x^2+1}\right)\mathrm{d}x = \frac{x^3}{3} - x + \underbrace{\tan^{-1}\left(x\right)}_{\text{by (8.26)}} + C$$

2. (a) Completing the square on the denominator (although integrating by partial fractions would be easier) of the given integrand; $\int \frac{1}{x\left(x-1\right)}\,\mathrm{d}x$ gives

$$x\left(x-1\right) = x^2 - x = \left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

We have

$$\int \frac{1}{x\left(x-1\right)}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$
$$= -\int \frac{\mathrm{d}x}{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} \qquad \left[\text{Multiplying by } \frac{-1}{-1}=1\right]$$
$$= \underbrace{-\frac{1}{\cancel{2}\left(1/\cancel{2}\right)}\ln\left|\frac{\frac{1}{2}+\left(x-\frac{1}{2}\right)}{\frac{1}{2}-\left(x-\frac{1}{2}\right)}\right|}_{\text{by (8.30)}} + C = C - \ln\left|\frac{x}{1-x}\right|$$

(b) Completing the square on the denominator (again using partial fractions in this case would be easier) of the given integrand;

$$x^2 - 7x + 10 = \left(x - \frac{7}{2}\right)^2 + 10 - \frac{49}{4}$$
$$= \left(x - \frac{7}{2}\right)^2 - \frac{9}{4} = \left(x - \frac{7}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

Putting this back into the integrand gives

$$\int \frac{1}{x^2 - 7x + 10}\,\mathrm{d}x = \int \frac{1}{\left(x - \frac{7}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,\mathrm{d}x$$
$$= -\int \frac{1}{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\,\mathrm{d}x$$
$$= \underbrace{-\frac{1}{\cancel{2}\left(\frac{3}{\cancel{2}}\right)} \ln\left|\frac{\frac{3}{2} + \left(x - \frac{7}{2}\right)}{\frac{3}{2} - \left(x - \frac{7}{2}\right)}\right|}_{\text{by (8.30)}} + C = C - \frac{1}{3}\ln\left|\frac{x-2}{5-x}\right|$$

(c) Multiplying the numerator and denominator of the integrand by $\frac{1}{3}$:

$$\int \frac{1}{2 - 3x^2}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\frac{2}{3} - x^2}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\left(\sqrt{\frac{2}{3}}\right)^2 - x^2}\,\mathrm{d}x$$

Applying the standard integral formula:

(8.30) $$\int \frac{\mathrm{d}u}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) \quad \text{or} \quad \int \frac{\mathrm{d}u}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$$

Gives

$$\frac{1}{3}\int \frac{1}{\left(\sqrt{\frac{2}{3}}\right)^2 - x^2}\,\mathrm{d}x = \frac{1}{3\sqrt{\frac{2}{3}}}\tanh^{-1}\left(\sqrt{\frac{3}{2}}x\right) + C = \frac{1}{\sqrt{6}}\tanh^{-1}\left(\sqrt{\frac{3}{2}}x\right) + C$$

(d) We are given the integral $\int \frac{1}{4x^2 + 4x + 5}\,\mathrm{d}x$. Taking out the 1/4 gives

$$\int \frac{1}{4x^2 + 4x + 5}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{x^2 + x + \frac{5}{4}}\,\mathrm{d}x$$

Completing the square on this last integrand:

$$x^2 + x + \frac{5}{4} = \left(x + \frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + 1^2$$

Substituting this back into the integrand yields

$$\int \frac{1}{4x^2 + 4x + 5}\, \mathrm{d}x = \frac{1}{4}\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + 1^2}\, \mathrm{d}x \underbrace{=}_{\text{by (8.26)}} \frac{1}{4}\tan^{-1}\left(x + \frac{1}{2}\right) + C$$

(e) We are given $\int \frac{1}{\sqrt{4x - 3 - x^2}}\, \mathrm{d}x$. Completing the square for the expression under the square root sign gives

$$4x - 3 - x^2 = -\left(x^2 - 4x + 3\right) = -\left(\left[x - 2\right]^2 - 1^2\right) = 1^2 - \left(x - 2\right)^2$$

Therefore the integral becomes

$$\int \frac{1}{\sqrt{4x - 3 - x^2}}\, \mathrm{d}x = \int \frac{1}{\sqrt{1^2 - \left(x - 2\right)^2}}\, \mathrm{d}x \underbrace{=}_{\text{by (8.25)}} \sin^{-1}\left(x - 2\right) + C$$

(f) We need to find the integral $\int \frac{1}{\sqrt{2 - 6x - 9x^2}}\, \mathrm{d}x$. Completing the square on the quadratic term under the square root:

$$\begin{aligned} 2 - 6x - 9x^2 &= -\left(9x^2 + 6x - 2\right) \\ &= -9\left(x^2 + \frac{6}{9}x - \frac{2}{9}\right) \\ &= -3^2\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{3}\right] = 3 - 3^2\left(x + \frac{1}{3}\right)^2 = 3 - \left(3x + 1\right)^2 \end{aligned}$$

Therefore we have

$$\begin{aligned} \int \frac{1}{\sqrt{2 - 6x - 9x^2}}\, \mathrm{d}x &= \int \frac{1}{\sqrt{3 - \left(3x + 1\right)^2}}\, \mathrm{d}x \\ &\underbrace{=}_{\text{Mulitply by } \frac{3}{3}=1} \frac{1}{3}\int \frac{3}{\sqrt{\left(\sqrt{3}\right)^2 - \left(3x + 1\right)^2}}\, \mathrm{d}x \underbrace{=}_{\text{by (8.25)}} \frac{1}{3}\sin^{-1}\left(\frac{\left(3x + 1\right)}{\sqrt{3}}\right) + C \end{aligned}$$

3. (a) We are asked to find $\int \frac{1}{1 - \cos\left(x\right)}\, \mathrm{d}x$. *Which trigonometric identity do we use to find this integral?*

(4.73) $$\cos\left(x\right)=\frac{1-t^2}{1+t^2} \text{ where } t=\tan\left(\frac{x}{2}\right)$$

Differentiating $t=\tan\left(\frac{x}{2}\right)$ we have

$$\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)=\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) \Rightarrow dx=\frac{2dt}{1+t^2} \quad \left[\begin{array}{l}\text{Because}\\ t=\tan\left(\frac{x}{2}\right)\end{array}\right]$$

Using this substitution in the given integral we have

$$\begin{aligned}\int\frac{1}{1-\cos\left(x\right)}\,dx&=\int\frac{1}{1-\frac{1-t^2}{1+t^2}}\left(\frac{2dt}{1+t^2}\right)\\&=\int\frac{2dt}{1+t^2-1+t^2} \quad \left[\text{Simplifying}\right]\\&=\int\frac{dt}{t^2}=\int t^{-2}dt=-\frac{1}{t}+C=C-\frac{1}{\tan\left(\frac{x}{2}\right)} \quad \left[\begin{array}{l}\text{Recall}\\ t=\tan\left(\frac{x}{2}\right)\end{array}\right]\end{aligned}$$

(b) This integral $\int\frac{1}{1+\sin\left(x\right)}\,dx$ is similar to part (a) but we use

(4.72) $$\sin\left(x\right)=\frac{2t}{1+t^2} \quad \text{where } t=\tan\left(\frac{x}{2}\right)$$

We have

$$\begin{aligned}\int\frac{1}{1+\sin\left(x\right)}\,dx&=\int\frac{1}{1+\frac{2t}{1+t^2}}\underbrace{\left(\frac{2dt}{1+t^2}\right)}_{\text{from part (a)}}\\&=\int\frac{2dt}{1+2t+t^2}\\&=2\int\frac{dt}{\left(t+1\right)^2}=2\int\left(t+1\right)^{-2}dt\\&=-\frac{2}{t+1}+C=C-\frac{2}{\tan\left(\frac{x}{2}\right)+1}\end{aligned}$$

(c) We need to find $\int\frac{1-\cos\left(x\right)}{1+\cos\left(x\right)}\,dx$. Using the substitution given in the solution of part (a) we have

$$\int \frac{1-\cos(x)}{1+\cos(x)}\,dx = \int \frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\left(\frac{2dt}{1+t^2}\right)$$

$$= \int \frac{1+t^2-(1-t^2)}{1+t^2+1-t^2}\left(\frac{2dt}{1+t^2}\right) = \int \frac{\cancel{2}t^2}{\cancel{2}}\left(\frac{2dt}{1+t^2}\right) = \int \frac{2t^2}{1+t^2}dt$$

Applying long division to the last integrand in the above gives

$$\frac{2t^2}{1+t^2} = 2 - \frac{2}{1+t^2}$$

Now integrating this we have

$$\int \frac{1-\cos(x)}{1+\cos(x)}\,dx = \int \frac{2t^2}{1+t^2}\,dt = \int\left(2-\frac{2}{1+t^2}\right)dt$$

$$= 2t - 2\tan^{-1}(t) + C$$

$$= 2\tan\left(\frac{x}{2}\right) - 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\right) + C$$

$$= 2\tan\left(\frac{x}{2}\right) - x + C \qquad \left[\text{Because } \tan^{-1}(\tan(\theta)) = \theta\right]$$

(d) We are asked to integrate $\int \left[\tan^2(x) + \tan^4(x)\right] dx$. Rewriting the integrand we have

$$\tan^2(x) + \tan^4(x) = \tan^2(x)\left[1+\tan^2(x)\right] = \tan^2(x)\sec^2(x)$$

Let $u = \tan(x)$ then differentiating this gives

$$\frac{du}{dx} = \sec^2(x) \;\Rightarrow\; dx = \frac{du}{\sec^2(x)}$$

Therefore we have

$$\int \left[\tan^2(x) + \tan^4(x)\right] dx = \int u^2 \cancel{\sec^2(x)} \frac{du}{\cancel{\sec^2(x)}}$$

$$= \frac{u^3}{3} + C = \frac{\tan^3(x)}{3} + C$$

(e) We are asked to find $\int \frac{\cos(2x)}{1+\sin(x)\cos(x)}\,dx$. Using

(4.53) $$\sin(2x) = 2\sin(x)\cos(x)$$

Therefore

$$\int \frac{\cos(2x)}{1+\sin(x)\cos(x)}\,\mathrm{d}x = \int \frac{\cos(2x)}{1+\frac{1}{2}\sin(2x)}\,\mathrm{d}x \underbrace{=}_{\substack{\text{Multiplying numerator}\\ \text{and denominator by 2}}} \int \frac{2\cos(2x)}{2+\sin(2x)}\,\mathrm{d}x$$

We have

$$\int \frac{\cos(2x)}{1+\sin(x)\cos(x)}\,\mathrm{d}x = \int \frac{2\cos(2x)}{2+\sin(2x)}\,\mathrm{d}x = \ln\left|2+\sin(2x)\right| + C$$

(f) We need to use an appropriate trigonometric identity in order to find $\int \sin(2x)\sin(5x)\,\mathrm{d}x$. *Which one?*

(4.59) $$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

Applying this to the given integrand yields

$$\begin{aligned}\sin(2x)\sin(5x) &= \frac{1}{2}\left[\cos(2x-5x) - \cos(2x+5x)\right]\\ &= \frac{1}{2}\left[\cos(-3x) - \cos(7x)\right]\\ &= \frac{1}{2}\left[\cos(3x) - \cos(7x)\right] \qquad \left[\text{Because } \cos(-\theta) = \cos(\theta)\right]\end{aligned}$$

It is much easier to integrate this:

$$\begin{aligned}\int \sin(2x)\sin(5x)\,\mathrm{d}x &= \frac{1}{2}\int \left[\cos(3x) - \cos(7x)\right]\mathrm{d}x\\ &= \frac{1}{2}\left[\frac{\sin(3x)}{3} - \frac{\sin(7x)}{7}\right] + C\end{aligned}$$

(g) We are asked to find $\int \cos(x)\cos(2x)\cos(3x)\,\mathrm{d}x$. Using the identity:

(4.58) $$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

On $\cos(2x)\cos(3x)$ gives

$$\begin{aligned}\cos(2x)\cos(3x) &= \frac{1}{2}\left[\cos(2x+3x) + \cos(2x-3x)\right]\\ &= \frac{1}{2}\left[\cos(5x) + \cos(-x)\right] = \frac{1}{2}\left[\cos(5x) + \cos(x)\right]\end{aligned}$$

Multiplying this by $\cos(x)$ yields

$$\cos(x)\cos(2x)\cos(3x) = \frac{1}{2}\left[\cos(x)\cos(5x) + \cos^2(x)\right] \qquad (*)$$

Applying identity (4.58) again and using

(4.67) $$\cos^2(x) = \frac{1}{2}\left[1 + \cos(2x)\right]$$

On (*) gives us

$$\begin{aligned}\cos(x)\cos(2x)\cos(3x) &= \frac{1}{2}\left[\cos(x)\cos(5x)+\cos^2(x)\right]\\ &= \frac{1}{2}\left[\frac{1}{2}\left[\cos(6x)+\cos(-4x)\right]+\frac{1}{2}\left(1+\cos(2x)\right)\right]\\ &= \frac{1}{4}\left[\cos(6x)+\cos(4x)+1+\cos(2x)\right]\end{aligned}$$

Now integrating the right hand side of this is much easier than integrating the left hand side:

$$\begin{aligned}\int \cos(x)\cos(2x)\cos(3x)\,\mathrm{d}x &= \frac{1}{4}\int\left[\cos(6x)+\cos(4x)+1+\cos(2x)\right]\mathrm{d}x\\ &= \frac{1}{4}\left[\frac{\sin(6x)}{6}+\frac{\sin(4x)}{4}+x+\frac{\sin(2x)}{2}\right]+C\end{aligned}$$

(h) We are given $\int \frac{\sin^3(x)}{\sqrt{\cos(x)}}\,\mathrm{d}x$. Writing

$$\sin^3(x)=\sin(x)\sin^2(x)=\sin(x)\left[1-\cos^2(x)\right]$$

Let $u=\cos(x)$ then

$$\mathrm{d}u=-\sin(x)\,\mathrm{d}x \quad\Rightarrow\quad \mathrm{d}x=-\frac{\mathrm{d}u}{\sin(x)}$$

Using this substitution we have

$$\begin{aligned}\int\frac{\sin^3(x)}{\sqrt{\cos(x)}}\,\mathrm{d}x &= \int\frac{\sin(x)\left[1-\cos^2(x)\right]}{\sqrt{\cos(x)}}\,\mathrm{d}x\\ &= -\int\frac{\cancel{\sin(x)}\left[1-u^2\right]}{\sqrt{u}}\frac{\mathrm{d}u}{\cancel{\sin(x)}}\\ &= -\left[\int u^{-\frac{1}{2}}\mathrm{d}u-\int u^{\frac{3}{2}}\mathrm{d}u\right]\\ &= -\left[2u^{\frac{1}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right]+C\\ &= 2\left[\frac{\cos^{\frac{5}{2}}(x)}{5}-\sqrt{\cos(x)}\right]+C = 2\sqrt{\cos(x)}\left[\frac{\cos^5(x)}{5}-1\right]+C\end{aligned}$$

(i) We need to find $\int\frac{1}{\cos^4(x)}\mathrm{d}x$. Recall that $\frac{1}{\cos(x)}=\sec(x)$ so

$$\frac{1}{\cos^4(x)} = \sec^4(x) = \sec^2(x)\sec^2(x) = \sec^2(x)\left[1+\tan^2(x)\right]$$

Let $u = \tan(x)$ then

$$\mathrm{d}u = \sec^2(x)\,\mathrm{d}x \;\Rightarrow\; \mathrm{d}x = \frac{\mathrm{d}u}{\sec^2(x)}$$

Using this substitution $u = \tan(x)$ gives

$$\begin{aligned}\int \frac{1}{\cos^4(x)}\mathrm{d}x &= \int \sec^2(x)\left[1+\tan^2(x)\right]\mathrm{d}x \\ &= \int \cancel{\sec^2(x)}\left[1+u^2\right]\frac{\mathrm{d}u}{\cancel{\sec^2(x)}} \\ &= u + \frac{u^3}{3} + C = \tan(x) + \frac{\tan^3(x)}{3} + C\end{aligned}$$

(j) We are given $\int \tan^4(x)\,\mathrm{d}x$. Using trigonometric identities we have

$$\begin{aligned}\tan^4(x) &= \tan^2(x)\tan^2(x) \\ &= \tan^2(x)\left[\sec^2(x)-1\right] \\ &= \tan^2(x)\sec^2(x) - \tan^2(x) \quad \left[\text{Expanding}\right]\end{aligned}$$

Therefore we have

$$\int \tan^4(x)\,\mathrm{d}x = \int \tan^2(x)\sec^2(x)\,\mathrm{d}x - \int \tan^2(x)\,\mathrm{d}x \quad (*)$$

Finding each of these integrals separately:

$$\begin{aligned}\int \tan^2(x)\,\mathrm{d}x &= \int \frac{\sin^2(x)}{\cos^2(x)}\,\mathrm{d}x \\ &= \int \frac{1-\cos^2(x)}{\cos^2(x)}\,\mathrm{d}x \\ &= \int \frac{1}{\cos^2(x)}\,\mathrm{d}x - \int \frac{\cos^2(x)}{\cos^2(x)}\,\mathrm{d}x \\ &= \int \sec^2(x)\,\mathrm{d}x - \int 1\,\mathrm{d}x \\ &= \tan(x) - x\end{aligned}$$

(Ignoring the integration constant and adding this at the end.)

Determining the other integral in (*) which is $\int \tan^2(x)\sec^2(x)\,\mathrm{d}x$:

Let $u = \tan(x)$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2(x) \;\Rightarrow\; \mathrm{d}x = \frac{\mathrm{d}u}{\sec^2(x)}$. We have

$$\int \tan^2(x)\sec^2(x)\,\mathrm{d}x = \int u^2 \cancel{\sec^2(x)}\frac{\mathrm{d}u}{\cancel{\sec^2(x)}}$$
$$= \frac{u^3}{3} = \frac{\tan^3(x)}{3}$$

(Again ignoring the constant of integration for the time being.)

Substituting each of these $\int \tan^2(x)\,\mathrm{d}x = \tan(x) - x$ and

$\int \tan^2(x)\sec^2(x)\,\mathrm{d}x = \dfrac{\tan^3(x)}{3}$ into (*) gives

$$\int \tan^4(x)\,\mathrm{d}x = \int \tan^2(x)\sec^2(x)\,\mathrm{d}x - \int \tan^2(x)\,\mathrm{d}x$$
$$= \frac{\tan^3(x)}{3} - \tan(x) + x + C$$

(k) We need to determine $\int \sin^5(x)\,\mathrm{d}x$. We can rewrite the integrand as

$$\sin^5(x) = \sin(x)\sin^2(x)\sin^2(x) = \sin(x)\left[1-\cos^2(x)\right]\left[1-\cos^2(x)\right]$$

Let $u = \cos(x)$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x) \Rightarrow \mathrm{d}x = -\dfrac{\mathrm{d}u}{\sin(x)}$. We have

$$\int \sin^5(x)\,\mathrm{d}x = \int \sin(x)\left[1-\cos^2(x)\right]\left[1-\cos^2(x)\right]\mathrm{d}x$$
$$= -\int \cancel{\sin(x)}\left[1-u^2\right]\left[1-u^2\right]\frac{\mathrm{d}u}{\cancel{\sin(x)}}$$
$$= -\int \left(1-2u^2+u^4\right)\mathrm{d}u$$
$$= -\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right] + C = C - \cos(x) + \frac{2\cos^3(x)}{3} - \frac{\cos^5(x)}{5}$$

(l) We are asked to find $\int \dfrac{1}{\sin^6(x)}\,\mathrm{d}x$. We can rewrite the integrand as

$$\frac{1}{\sin^6(x)} = \cos ec^6(x) = \cos ec^2(x)\cos ec^4(x)$$

We use the following trigonometric identity:

(4.66) $$\cos ec^2(x) = 1 + \cot^2(x)$$

Substituting this into the above integrand gives

$$\cos ec^2(x)\cos ec^4(x) = \cos ec^2(x)\left[1+\cot^2(x)\right]^2$$

Let $u = \cot(x)$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} \underbrace{=}_{\text{by (6.24)}} -\cos ec^2(x) \Rightarrow \mathrm{d}x = -\dfrac{\mathrm{d}u}{\cos ec^2(x)}$.

Therefore

$$\begin{aligned}\int \frac{1}{\sin^6(x)}\,dx &= \int \operatorname{cosec}^2(x)\left[1+\cot^2(x)\right]^2\,dx\\ &= -\int \cancel{\operatorname{cosec}^2(x)}\left[1+u^2\right]^2\,\frac{du}{\cancel{\operatorname{cosec}^2(x)}}\\ &= -\int\left(1+2u^2+u^4\right)du\\ &= -\left[u+\frac{2u^3}{3}+\frac{u^5}{5}\right]+C\\ &= C-\left[\cot(x)+\frac{2\cot^3(x)}{3}+\frac{\cot^5(x)}{5}\right]\end{aligned}$$

4. We need to use integration by parts in each case.

(a) We are given the integral $\int x\sin(2x)\,dx$. Let

$$u = x \qquad v' = \sin(2x)$$
$$u' = 1 \qquad v = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x)$$

Applying the integration by parts formula we have

$$\begin{aligned}\int x\sin(2x)\,dx &= uv - \int u'v\,dx\\ &= -\frac{1}{2}x\cos(2x)+\frac{1}{2}\int\cos(2x)\,dx\\ &= \frac{1}{2}\left[-x\cos(2x)+\frac{1}{2}\sin(2x)\right]+C = \frac{1}{4}\left[\sin(2x)-2x\cos(2x)\right]+C\end{aligned}$$

(b) We are asked to find $\int 3^x x\,dx$. Nominating u and v' as follows:

$$u = x \qquad v' = 3^x$$
$$u' = 1 \qquad v = \int 3^x\,dx \underbrace{=}_{\text{by (8.4)}} \frac{1}{\ln(3)}3^x$$

Substituting these into the integration by parts formula:

$$\begin{aligned}\int 3^x x\,dx &= uv - \int u'v\,dx\\ &= \frac{1}{\ln(3)}3^x x - \frac{1}{\ln(3)}\int 3^x\,dx\\ &= \frac{1}{\ln(3)}3^x x - \frac{1}{\ln^2(3)}3^x + C = \frac{1}{\ln^2(3)}3^x\left[x\ln(3)-1\right]+C\end{aligned}$$

(c) We are asked to find $\int x\tan^{-1}(x)\,dx$. Let

$$u = \tan^{-1}(x) \qquad v' = x$$
$$u' = \frac{1}{1+x^2} \qquad v = \int x \, dx = \frac{1}{2}x^2$$

We have

$$\begin{aligned}\int x\tan^{-1}(x)\,dx &= uv - \int u'v\,dx \\ &= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}\left[x^2\tan^{-1}(x) - \int \underbrace{\left(1 - \frac{1}{1+x^2}\right)}_{\text{by long division}} dx\right] \\ &= \frac{1}{2}\left[x^2\tan^{-1}(x) - x + \tan^{-1}(x)\right] + C\end{aligned}$$

(d) We are given $\int \cos^{-1}(x)\,dx$. We can rewrite the integrand as

$$\cos^{-1}(x) = \cos^{-1}(x) \times 1$$

Let $u = \cos^{-1}(x)$ and $v' = 1$. Then

$$u' = \left[\cos^{-1}(x)\right]'$$

How do we differentiate this function?

Let $y = \cos^{-1}(x)$ then $\cos(y) = x$ and so

$$-\sin(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin(y)} \qquad (*)$$

By the fundamental trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ we have

$$\sin(y) = \sqrt{1 - \cos^2(y)}$$

Since $y = \cos^{-1}(x)$ so

$$\sin(y) = \sqrt{1 - \cos^2\left(\cos^{-1}(x)\right)} = \sqrt{1 - x^2}$$

Putting this $\sin(y) = \sqrt{1-x^2}$ into (*) gives

$$u' = \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

The various parts of the integration by parts formula are given by

$$u = \cos^{-1}(x) \text{ and } v' = 1$$
$$u' = -\frac{1}{\sqrt{1-x^2}} \text{ and } v = x$$

We have

$$\int \cos^{-1}(x)\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$
$$= x\cos^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \qquad (**)$$

The last term on the right hand side of (**) can be found by differentiating:

$$\left(\sqrt{1-x^2}\right)' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

Therefore $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\sqrt{1-x^2}$.

Substituting this $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\sqrt{1-x^2}$ into (**) gives

$$\int \cos^{-1}(x)\,\mathrm{d}x = x\cos^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$= x\cos^{-1}(x) - \sqrt{1-x^2} + C$$

(e) We are asked to find $\int \frac{\log_{10}(x)}{x^3}\,\mathrm{d}x$. We can rewrite the integrand as

$$\frac{\log_{10}(x)}{x^3} = x^{-3}\log_{10}(x)$$

Let $u = \log_{10}(x)$ and $v' = x^{-3}$. In order to find $u' = \left[\log_{10}(x)\right]'$ we have to change the base of the logarithm to the natural logarithm because our table of derivatives *only* gives the derivative of the natural logarithm.

$$u = \log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

Now we have

$$u' = \left[\frac{\ln(x)}{\ln(10)}\right]' = \frac{1}{x\ln(10)} \text{ and } v = \int x^{-3}\,\mathrm{d}x = -\frac{1}{2x^2}$$

Therefore applying the integration by parts formula we have

$$\begin{aligned}\int \frac{\log_{10}(x)}{x^3}\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ &= -\frac{\ln(x)}{2x^2\ln(10)} + \frac{1}{\ln(10)}\int \frac{1}{2x^3}\,\mathrm{d}x \\ &= -\frac{\ln(x)}{2x^2\ln(10)} + \frac{1}{2\ln(10)}\left[-\frac{1}{2x^2}\right] + C \\ &= -\frac{1}{4x^2\ln(10)}\left[2\ln(x)+1\right] + C = C - \frac{1}{4x^2\ln(10)}\left[\ln(x^2)+1\right]\end{aligned}$$

(f) We are asked to find $\int \frac{\sin^{-1}(\sqrt{x})}{\sqrt{1-x}}\,\mathrm{d}x$. Let $u = \sin^{-1}(\sqrt{x})$ then

$$\sin(u) = x^{\frac{1}{2}} \underbrace{\Rightarrow}_{\text{Differentiating}} \cos(u)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}\cos(u)} \qquad (\dagger)$$

From the fundamental trigonometric identity we have

$$\cos(u) = \sqrt{1-\sin^2(u)} = \sqrt{1-\sin^2\left[\sin^{-1}(\sqrt{x})\right]} = \sqrt{1-x}$$

Substituting this $\cos(u) = \sqrt{1-x}$ into (†) gives

$$u' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

We also have

$$v' = \frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}} \Rightarrow v = \int (1-x)^{-\frac{1}{2}}\,\mathrm{d}x = -2(1-x)^{\frac{1}{2}} = -2\sqrt{1-x}$$

Substituting various components $u = \sin^{-1}(\sqrt{x})$, $u' = \frac{1}{2\sqrt{x}\sqrt{1-x}}$ and

$v = -2\sqrt{1-x}$ into the integration by parts formula gives

$$\begin{aligned}\int \frac{\sin^{-1}(\sqrt{x})}{\sqrt{1-x}}\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ &= -2\sqrt{1-x}\sin^{-1}(\sqrt{x}) + \cancel{2}\int \frac{1}{\cancel{2}\sqrt{x}\,\cancel{\sqrt{1-x}}}\cancel{\sqrt{1-x}}\,\mathrm{d}x \\ &= -2\sqrt{1-x}\sin^{-1}(\sqrt{x}) + \int x^{-\frac{1}{2}}\,\mathrm{d}x \\ &= -2\sqrt{1-x}\sin^{-1}(\sqrt{x}) + 2\sqrt{x} + C \\ &= 2\left[\sqrt{x} - \sqrt{1-x}\sin^{-1}(\sqrt{x})\right] + C\end{aligned}$$

(g) We are given $\int x^2\ln(1+x)\,\mathrm{d}x$. Let

$$u = \ln(1+x) \qquad v' = x^2$$
$$u' = \frac{1}{1+x} \qquad v = \frac{x^3}{3}$$

Putting these into the integration by parts formula gives

$$\int x^2 \ln(1+x)\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$
$$= \frac{x^3}{3}\ln(1+x) - \frac{1}{3}\int \frac{x^3}{1+x} \qquad (*)$$

Applying long division to the last integrand on the right hand side:

$$\begin{array}{r|l} & \;\; x^2 - x + 1 \\ \hline 1+x & \;\; x^3 \\ & -(x^3 + x^2) \\ \hline & 0 \;- x^2 \\ & \quad -(-x^2 - x) \\ \hline & \qquad 0 + x \\ & \qquad -(x+1) \\ \hline & \qquad\quad 0 - 1 \end{array}$$

Therefore the integrand can be rewritten as

$$\frac{x^3}{1+x} = x^2 - x + 1 - \frac{1}{1+x}$$

Now integrating this is much easier:

$$\int \frac{x^3}{1+x}\,\mathrm{d}x = \int \left(x^2 - x + 1 - \frac{1}{1+x}\right)\mathrm{d}x$$
$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x|$$

Substituting this into (*) and adding the constant of integration we have

$$\int x^2 \ln(1+x)\,\mathrm{d}x = \frac{x^3}{3}\ln(1+x) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x|\right] + C$$
$$= \frac{x^3}{9}\left[3\ln(1+x) - 1\right] + \frac{1}{3}\left[\frac{x^2}{2} - x + \ln|1+x|\right] + C$$

(h) We need to find $\int \cos(\ln(x))\,\mathrm{d}x$. We can write the integrand as

$$\cos(\ln(x)) \times 1$$

Let $u = \cos(\ln(x))$ and $v' = 1$ then

$$u' = -\frac{\sin(\ln(x))}{x} \quad \text{and} \quad v = x$$

Therefore we have

$$\begin{aligned}\int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ &= x\cos\left(\ln\left(x\right)\right) + \int \sin\left(\ln\left(x\right)\right)\,\mathrm{d}x \qquad (*)\end{aligned}$$

Using integration by parts again to find the last integral on the right hand side by using symmetry we have:

$$\int \sin\left(\ln\left(x\right)\right)\,\mathrm{d}x = x\sin\left(\ln\left(x\right)\right) - \int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x$$

Putting this into (*) gives

$$\int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x = x\cos\left(\ln\left(x\right)\right) + x\sin\left(\ln\left(x\right)\right) - \int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x$$

Adding $\int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x$ to both sides gives

$$2\int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x = x\cos\left(\ln\left(x\right)\right) + x\sin\left(\ln\left(x\right)\right)$$

Multiplying both sides by ½ and adding the constant of integration gives

$$\int \cos\left(\ln\left(x\right)\right)\,\mathrm{d}x = \frac{1}{2}\left[x\cos\left(\ln\left(x\right)\right) + x\sin\left(\ln\left(x\right)\right)\right] + C$$

(i) We need to find $\int \sqrt{a^2 + x^2}\,\mathrm{d}x$ by using integration by parts.

Rewriting the integrand as

$$\sqrt{a^2 + x^2} = \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} = \frac{a^2}{\sqrt{a^2 + x^2}} + \frac{x^2}{\sqrt{a^2 + x^2}}$$

This looks more complex than our given integrand but it is easier to integrate the right hand side:

$$\begin{aligned}\int \sqrt{a^2 + x^2}\,\mathrm{d}x &= \int \frac{a^2}{\sqrt{a^2 + x^2}}\,\mathrm{d}x + \int \frac{x^2}{\sqrt{a^2 + x^2}}\,\mathrm{d}x \\ &= \underbrace{a^2\sinh^{-1}\left(\frac{x}{a}\right)}_{\text{by (8.28)}} + \int \frac{x^2}{\sqrt{a^2 + x^2}}\,\mathrm{d}x \qquad (\dagger)\end{aligned}$$

Applying integration by parts to find the last integral $\int x\frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x$:

$$\begin{aligned} u &= x & v' &= \frac{x}{\sqrt{a^2 + x^2}} \\ u' &= 1 & v &= \int \frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x \end{aligned}$$

We need to find $v = \int \frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x$. *How?*

By substitution with $r = a^2 + x^2 \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}x} = 2x$. Therefore

$$v = \int \frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x = \int \frac{\cancel{x}}{\sqrt{r}} \frac{\mathrm{d}r}{2\cancel{x}} = \frac{1}{2}\int r^{-\frac{1}{2}}\mathrm{d}r = \frac{\cancel{1}}{\cancel{2}} \frac{r^{\frac{1}{2}}}{\frac{\cancel{1}}{\cancel{2}}} = \sqrt{a^2 + x^2}$$

Putting this into the integration by parts formula gives

$$\begin{aligned}\int x \frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ &= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\,\mathrm{d}x\end{aligned}$$

Substituting this $\int x \frac{x}{\sqrt{a^2 + x^2}}\,\mathrm{d}x = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\,\mathrm{d}x$ into (†) gives

$$\int \sqrt{a^2 + x^2}\,\mathrm{d}x = a^2 \sinh^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\,\mathrm{d}x$$

Adding $\int \sqrt{a^2 + x^2}\,\mathrm{d}x$ to both sides gives

$$2\int \sqrt{a^2 + x^2}\,\mathrm{d}x = a^2 \sinh^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2 + x^2}$$

Dividing both sides by 2 and adding the constant of integration yields

$$\int \sqrt{a^2 + x^2}\,\mathrm{d}x = \frac{1}{2}\left[a^2 \sinh^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2 + x^2}\right] + C$$

(j) We are asked to find $\int \frac{x^2}{\sqrt{1 - x^2}}\,\mathrm{d}x$. We can write this as

$$\int x \frac{x}{\sqrt{1 - x^2}}\,\mathrm{d}x$$

We nominate u and v' as follows:

$$\begin{aligned} u &= x & v' &= \frac{x}{\sqrt{1 - x^2}} \\ u' &= 1 & v &= \int \frac{x}{\sqrt{1 - x^2}}\,\mathrm{d}x \end{aligned}$$

We need to use integration by substitution to find v:

$$v = \int \frac{x}{\sqrt{1 - x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{\cancel{x}}{\sqrt{r}} \frac{\mathrm{d}r}{\cancel{x}} = -\frac{1}{2}\int r^{-\frac{1}{2}}\mathrm{d}r = -\frac{\cancel{1}}{\cancel{2}} \frac{r^{\frac{1}{2}}}{\frac{\cancel{1}}{\cancel{2}}} = -\sqrt{1 - x^2}$$

Applying the integration by parts formula gives

$$\int x\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$
$$= -x\sqrt{1-x^2} + \int\sqrt{1-x^2}\,\mathrm{d}x \qquad (*)$$

We need to find the last integral on the right hand side of the above. Let $x=\sin\left(\theta\right)$ then $\mathrm{d}x=\cos\left(\theta\right)\mathrm{d}\theta$. We have

$$\int\sqrt{1-x^2}\,\mathrm{d}x = \int\sqrt{1-\sin^2\left(\theta\right)}\cos\left(\theta\right)\mathrm{d}\theta$$
$$= \int\cos^2\left(\theta\right)\mathrm{d}\theta$$
$$= \frac{1}{2}\int\left[1+\cos\left(2\theta\right)\right]\mathrm{d}\theta$$
$$= \frac{1}{2}\left[\theta+\frac{\sin\left(2\theta\right)}{2}\right]$$
$$= \frac{1}{2}\left[\theta+\frac{2\sin\left(\theta\right)\cos\left(\theta\right)}{2}\right]$$
$$= \frac{1}{2}\left[\theta+\sin\left(\theta\right)\cos\left(\theta\right)\right]$$

From above we have $x=\sin\left(\theta\right)$ so $\theta=\sin^{-1}\left(x\right)$. Substituting this into the above gives

$$\int\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left[\sin^{-1}\left(x\right)+\sin\left(\sin^{-1}\left(x\right)\right)\cos\left(\sin^{-1}\left(x\right)\right)\right]$$
$$= \frac{1}{2}\left[\sin^{-1}\left(x\right)+x\cos\left(\sin^{-1}\left(x\right)\right)\right]$$

Recall that $\cos\left(\alpha\right)=\sqrt{1-\sin^2\left(\alpha\right)}$ therefore

$$\cos\left(\sin^{-1}\left(x\right)\right)=\sqrt{1-\sin^2\left(\sin^{-1}\left(x\right)\right)}=\sqrt{1-x^2}$$

Substituting this into the above gives

$$\int\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left[\sin^{-1}\left(x\right)+x\cos\left(\sin^{-1}\left(x\right)\right)\right]$$
$$= \frac{1}{2}\left[\sin^{-1}\left(x\right)+x\sqrt{1-x^2}\right]$$

Putting this $\int\sqrt{1-x^2}\,\mathrm{d}x=\frac{1}{2}\left[\sin^{-1}\left(x\right)+x\sqrt{1-x^2}\right]$ into (*) yields

$$\int\frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = -x\sqrt{1-x^2}+\int\sqrt{1-x^2}\,\mathrm{d}x$$
$$= -x\sqrt{1-x^2}+\frac{1}{2}\left[\sin^{-1}\left(x\right)+x\sqrt{1-x^2}\right]+C$$
$$= \frac{1}{2}\left[\sin^{-1}\left(x\right)-x\sqrt{1-x^2}\right]+C$$

5. (a) We are given the integral $\int \frac{1}{1+\sqrt{x+1}}\,\mathrm{d}x$. Let $u=\sqrt{x+1}=\left(x+1\right)^{\frac{1}{2}}$.

Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}\left(x+1\right)^{-\frac{1}{2}}=\frac{1}{2\sqrt{x+1}} \quad \Rightarrow \quad \mathrm{d}x=2\sqrt{x+1}\mathrm{d}u=2u\ \mathrm{d}u$$

We have

$$\int \frac{1}{1+\sqrt{x+1}}\,\mathrm{d}x=\int \frac{1}{1+u}2u\ \mathrm{d}u=2\int \frac{u}{1+u}\mathrm{d}u$$

By using long division on the last integrand we have

$$\frac{u}{1+u}=1-\frac{1}{1+u}$$

Integrating this is straightforward:

$$\begin{aligned}\int \frac{1}{1+\sqrt{x+1}}\,\mathrm{d}x &= 2\int \frac{u}{1+u}\mathrm{d}u\\ &= 2\int\left(1-\frac{1}{1+u}\right)\mathrm{d}u\\ &= 2\left[u-\ln\left|1+u\right|\right]+C\\ &= 2\left[\sqrt{x+1}-\ln\left|1+\sqrt{x+1}\right|\right]+C \qquad \left[\text{Because } u=\sqrt{x+1}\right]\end{aligned}$$

(b) We need to find $\int \frac{x^3}{\sqrt{x-1}}\,\mathrm{d}x$. Let $u=x-1$ then $\mathrm{d}u=\mathrm{d}x$ and

$$x^3=\left(u+1\right)^3=u^3+3u^2+3u+1$$

Substituting these into the given integral yields

$$\begin{aligned}\int \frac{x^3}{\sqrt{x-1}}\,\mathrm{d}x &= \int \frac{u^3+3u^2+3u+1}{u^{\frac{1}{2}}}\,\mathrm{d}u\\ &= \int\left(u^{\frac{5}{2}}+3u^{\frac{3}{2}}+3u^{\frac{1}{2}}+u^{-\frac{1}{2}}\right)\mathrm{d}u\\ &= \frac{2}{7}u^{\frac{7}{2}}+\frac{6}{5}u^{\frac{5}{2}}+2u^{\frac{3}{2}}+2u^{\frac{1}{2}}+C\\ &= \frac{2}{7}\left(x-1\right)^{\frac{7}{2}}+\frac{6}{5}\left(x-1\right)^{\frac{5}{2}}+2\left(x-1\right)^{\frac{3}{2}}+2\left(x-1\right)^{\frac{1}{2}}+C\\ &= \left(x-1\right)^{\frac{1}{2}}\left[\frac{2}{7}\left(x-1\right)^3+\frac{6}{5}\left(x-1\right)^2+2\left(x-1\right)+2\right]+C\\ &= \frac{2}{35}\left(x-1\right)^{\frac{1}{2}}\left[5\left(x-1\right)^3+21\left(x-1\right)^2+35\left(x-1\right)+35\right]+C\end{aligned}$$

(c) We are given $\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,\mathrm{d}x$ and that $u=\sqrt[6]{x}=x^{\frac{1}{6}}$. Therefore

$$\sqrt{x}=x^{\frac{1}{2}}=x^{\frac{3}{6}}=\left(x^{\frac{1}{6}}\right)^3=u^3 \text{ and } \sqrt[3]{x}=x^{\frac{1}{3}}=x^{\frac{2}{6}}=\left(x^{\frac{1}{6}}\right)^2=u^2$$

Also from $u=\sqrt[6]{x}=x^{\frac{1}{6}}$ we have

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{6}x^{-\frac{5}{6}}=\frac{1}{6x^{\frac{5}{6}}} \Rightarrow \mathrm{d}x=6x^{\frac{5}{6}}\mathrm{d}u=6u^5\mathrm{d}u$$

Substituting these $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$ and $\mathrm{d}x=6u^5\mathrm{d}u$ into the given integral:

$$\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,\mathrm{d}x=\int \frac{u^3}{u^3-u^2}6u^5\mathrm{d}u=6\int \frac{u^8}{u^3-u^2}\mathrm{d}u \underbrace{=}_{\text{Cancelling}} 6\int \frac{u^6}{u-1}\mathrm{d}u \qquad (*)$$

Now we can either use long division or the following algebraic identity:

$$x^n-1=(x-1)\left(x^{n-1}+x^{n-2}+\cdots+x+1\right)$$

We have

$$u^6-1=(u-1)\left(u^5+u^4+u^3+u^2+u+1\right)$$

Therefore

$$u^6=(u-1)\left(u^5+u^4+u^3+u^2+u+1\right)+1$$

Putting this into (*) gives

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,\mathrm{d}x &= 6\int \frac{u^6}{u-1}\mathrm{d}u \\
&= 6\int \frac{(u-1)\left(u^5+u^4+u^3+u^2+u+1\right)+1}{u-1}\mathrm{d}u \\
&= 6\int\left[\left(u^5+u^4+u^3+u^2+u+1\right)+\frac{1}{u-1}\right]\mathrm{d}u \\
&= 6\left[\frac{1}{6}u^6+\frac{1}{5}u^5+\frac{1}{4}u^4+\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln\left|u-1\right|\right]+C \\
&= 6\left[\frac{1}{6}\left(x^{\frac{1}{6}}\right)^6+\frac{1}{5}\left(x^{\frac{1}{6}}\right)^5+\frac{1}{4}\left(x^{\frac{1}{6}}\right)^4+\frac{1}{3}\left(x^{\frac{1}{6}}\right)^3+\frac{1}{2}\left(x^{\frac{1}{6}}\right)^2+\left(x^{\frac{1}{6}}\right)+\ln\left|x^{\frac{1}{6}}-1\right|\right]+C \\
&= 6\left[\frac{1}{6}x+\frac{1}{5}x^{\frac{5}{6}}+\frac{1}{4}x^{\frac{2}{3}}+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{3}}+x^{\frac{1}{6}}+\ln\left|x^{\frac{1}{6}}-1\right|\right]+C \\
&= x+\frac{6}{5}x^{\frac{5}{6}}+\frac{3}{2}x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+6x^{\frac{1}{6}}+6\ln\left|x^{\frac{1}{6}}-1\right|+C
\end{aligned}$$

(d) We are asked to find $\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,\mathrm{d}x$. Let $u=e^x+1$ then

$$\mathrm{d}u = e^x \mathrm{d}x \;\Rightarrow\; \mathrm{d}x = \frac{\mathrm{d}u}{e^x}$$

We have

$$\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,\mathrm{d}x = \int \frac{e^x e^x}{\sqrt[4]{u}}\frac{\mathrm{d}u}{e^x} \qquad \left[\text{Because } e^{2x} = \left(e^x\right)^2 = e^x e^x\right]$$

$$= \int \frac{e^x}{\sqrt[4]{u}}\mathrm{d}u = \int \frac{u-1}{u^{\frac{1}{4}}}\,\mathrm{d}u$$

$$= \int \frac{u}{u^{\frac{1}{4}}}\mathrm{d}u - \int \frac{1}{u^{\frac{1}{4}}}\mathrm{d}u = \int u^{\frac{3}{4}}\mathrm{d}u - \int u^{-\frac{1}{4}}\mathrm{d}u = \frac{4}{7}u^{\frac{7}{4}} - \frac{4}{3}u^{\frac{3}{4}} + C$$

From above we have $u = e^x + 1$ so

$$\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,\mathrm{d}x = \frac{4}{7}\left(e^x+1\right)^{\frac{7}{4}} - \frac{4}{3}\left(e^x+1\right)^{\frac{3}{4}} + C$$

$$= \frac{4}{21}\left(e^x+1\right)^{\frac{3}{4}}\left[3\left(e^x+1\right)-7\right] + C$$

$$= \frac{4}{21}\left(e^x+1\right)^{\frac{3}{4}}\left[3e^x - 4\right] + C$$

(e) We are asked to find $\int \frac{\ln\left(\tan\left(x\right)\right)}{\sin\left(x\right)\cos\left(x\right)}\,\mathrm{d}x$. Let $u = \tan\left(x\right)$ then differentiating this gives

$$\mathrm{d}u = \sec^2\left(x\right)\,\mathrm{d}x = \frac{1}{\cos^2\left(x\right)}\mathrm{d}x \quad\Rightarrow\quad \mathrm{d}x = \cos^2\left(x\right)\mathrm{d}u$$

Therefore

$$\int \frac{\ln\left(\tan\left(x\right)\right)}{\sin\left(x\right)\cos\left(x\right)}\,\mathrm{d}x = \int \frac{\ln\left(u\right)}{\sin\left(x\right)\cancel{\cos\left(x\right)}}\cos^{\cancel{2}}\left(x\right)\mathrm{d}u$$

$$= \int \frac{\ln\left(u\right)\cos\left(x\right)}{\sin\left(x\right)}\,\mathrm{d}u = \int \frac{\ln\left(u\right)}{\tan\left(x\right)}\,\mathrm{d}u = \int \frac{\ln\left(u\right)}{u}\,\mathrm{d}u$$

Since the derivative of $\ln\left(u\right)$ is $\frac{1}{u}$ so

$$\int \frac{\ln\left(\tan\left(x\right)\right)}{\sin\left(x\right)\cos\left(x\right)}\,\mathrm{d}x = \int \frac{\ln\left(u\right)}{u}\,\mathrm{d}u = \frac{1}{2}\ln^2\left(u\right) + C$$

Substituting the above $u = \tan\left(x\right)$ into this gives

$$\int \frac{\ln\left(\tan\left(x\right)\right)}{\sin\left(x\right)\cos\left(x\right)}\,\mathrm{d}x = \frac{1}{2}\ln^2\left(\tan\left(x\right)\right) + C$$

(f) We are asked to find $\int \frac{\sqrt{1+\ln(x)}}{x\ln(x)}\,\mathrm{d}x$. Let $u = 1+\ln(x)$ then

$$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \quad \Rightarrow \quad \mathrm{d}x = x\,\mathrm{d}u$$

Using this substitution we have

$$\int \frac{\sqrt{1+\ln(x)}}{x\ln(x)}\,\mathrm{d}x = \int \frac{\sqrt{u}}{\not{x}\ln(x)}\not{x}\,\mathrm{d}u = \int \frac{\sqrt{u}}{u-1}\,\mathrm{d}u$$

We can write the denominator of the last integrand as

$$u-1 = \left(\sqrt{u}-1\right)\left(\sqrt{u}+1\right)$$

Hence the integral can be rewritten as

$$\int \frac{\sqrt{u}}{u-1}\,\mathrm{d}u = \int \frac{\sqrt{u}}{\left(\sqrt{u}-1\right)\left(\sqrt{u}+1\right)}\,\mathrm{d}u$$

Let $v = \sqrt{u}$. Differentiating this gives

$$\mathrm{d}v = \frac{1}{2\sqrt{u}}\mathrm{d}u \quad \Rightarrow \quad \mathrm{d}u = 2\sqrt{u}\,\mathrm{d}v = 2v\,\mathrm{d}v$$

Using this substitution gives

$$\int \frac{\sqrt{u}}{u-1}\,\mathrm{d}u = \int \frac{\sqrt{u}}{\left(\sqrt{u}-1\right)\left(\sqrt{u}+1\right)}\,\mathrm{d}u = \int \frac{v}{(v-1)(v+1)}2v\,\mathrm{d}v = 2\int \frac{v^2}{v^2-1}\mathrm{d}v$$

We can rewrite this integrand $\frac{v^2}{v^2-1}$ as

$$\frac{v^2}{v^2-1} = 1 + \frac{1}{v^2-1}$$

Integrating this is much easier, so we have

$$\begin{aligned}
\int \frac{\sqrt{u}}{u-1}\,\mathrm{d}u &= 2\int \frac{v^2}{v^2-1}\mathrm{d}v \\
&= 2\int \left(1+\frac{1}{v^2-1}\right)\mathrm{d}v \\
&= 2\left[\int 1\mathrm{d}v - \int \frac{1}{1-v^2}\,\mathrm{d}v\right] \\
&= 2\left[v - \underbrace{\frac{1}{2}\ln\left|\frac{1+v}{1-v}\right|}_{\text{by (8.30)}}\right] + C = 2v - \ln\left|\frac{1+v}{1-v}\right| + C
\end{aligned}$$

Recall that $v=\sqrt{u}$ and $u=1+\ln\left(x\right)$ therefore

$$v=\sqrt{1+\ln\left(x\right)}$$

Substituting this into the above gives

$$\begin{aligned}\int\frac{\sqrt{1+\ln\left(x\right)}}{x\ln\left(x\right)}\,\mathrm{d}x&=\int\frac{\sqrt{u}}{u-1}\,\mathrm{d}u\\&=2v-\ln\left|\frac{1+v}{1-v}\right|+C\\&=2\sqrt{1+\ln\left(x\right)}-\ln\left|\frac{1+\sqrt{1+\ln\left(x\right)}}{1-\sqrt{1+\ln\left(x\right)}}\right|+C\end{aligned}$$

(g) We need to find $\int\frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x$. Let $x=a\sinh\left(u\right)$ then

$$\mathrm{d}x=a\cosh\left(u\right)\mathrm{d}u$$

Transforming the given integral

$$\begin{aligned}\int\frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x&=\int\frac{1}{a^{\not 2}\sinh^2\left(u\right)\sqrt{a^2\sinh^2\left(u\right)+a^2}}\,\not a\cosh\left(u\right)\mathrm{d}u\\&=\frac{1}{a}\int\frac{1}{a\sinh^2\left(u\right)\underbrace{\sqrt{\sinh^2\left(u\right)+1}}_{\cosh\left(u\right)}}\cosh\left(u\right)\mathrm{d}u\\&=\frac{1}{a^2}\int\frac{1}{\sinh^2\left(u\right)\cancel{\cosh\left(u\right)}}\cancel{\cosh\left(u\right)}\mathrm{d}u\\&=\frac{1}{a^2}\int\frac{1}{\sinh^2\left(u\right)}\,\mathrm{d}u\qquad(*)\end{aligned}$$

Recall that $\sinh\left(u\right)=\frac{e^u-e^{-u}}{2}$, therefore

$$\sinh^2\left(u\right)=\left(\frac{e^u-e^{-u}}{2}\right)^2=\left(\frac{e^u-\frac{1}{e^u}}{2}\right)^2=\frac{\left(e^{2u}-1\right)^2}{4e^{2u}}$$

Substituting this into (*) gives

$$\begin{aligned}\int\frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x&=\frac{1}{a^2}\int\frac{1}{\sinh^2\left(u\right)}\,\mathrm{d}u\\&=\frac{1}{a^2}\int\frac{4e^{2u}}{\left(e^{2u}-1\right)^2}\,\mathrm{d}u\end{aligned}$$

Let $v = e^{2u} - 1$ then $\frac{dv}{du} = 2e^{2u} \quad \Rightarrow \quad du = \frac{dv}{2e^{2u}}$. So we have

$$\int \frac{1}{x^2\sqrt{x^2+a^2}}\,dx = \frac{1}{a^2}\int \frac{4e^{2u}}{\left(e^{2u}-1\right)^2}\,du$$

$$= \frac{1}{a^2}\int \frac{4\cancel{e^{2u}}}{v^2}\frac{dv}{2\cancel{e^{2u}}} = \frac{2}{a^2}\int v^{-2}dv = -\frac{2}{a^2 v} + C$$

Remember $v = e^{2u} - 1$ and $x = a\sinh\left(u\right)$. From this $x = a\sinh\left(u\right)$ we have

$$\sinh^{-1}\left(\frac{x}{a}\right) = u \quad \text{which implies } v = e^{2u} - 1 = e^{2\sinh^{-1}\left(\frac{x}{a}\right)} - 1$$

Hence

$$\int \frac{1}{x^2\sqrt{x^2+a^2}}\,dx = -\frac{2}{a^2 v} + C = C - \frac{2}{a^2\left[e^{2\sinh^{-1}\left(\frac{x}{a}\right)} - 1\right]}$$

6. (a) Converting the integrand of $\int \frac{x^2+1}{x\left(x^2-1\right)}\,dx$ into partial fractions:

$$\frac{x^2+1}{x\left(x^2-1\right)} = \frac{x^2+1}{x\left(x-1\right)\left(x+1\right)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Using the cover up rule to find A, B and C gives

$$A = \frac{0^2+1}{\left(0-1\right)\left(0+1\right)} = -1$$

$$B = \frac{1^2+1}{1\left(1+1\right)} = 1$$

$$C = \frac{\left(-1\right)^2+1}{-1\left(\left(-1\right)-1\right)} = 1$$

Therefore the integrand is given by

$$\frac{x^2+1}{x\left(x^2-1\right)} = -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

We have

$$\int \frac{x^2+1}{x\left(x^2-1\right)}\,\mathrm{d}x = -\int \frac{1}{x}\,\mathrm{d}x + \int \frac{1}{x-1}\,\mathrm{d}x + \int \frac{1}{x+1}\,\mathrm{d}x$$
$$= -\ln\left|x\right| + \ln\left|x-1\right| + \ln\left|x+1\right| + C$$
$$= \ln\left|\frac{x^2-1}{x}\right| + C \qquad \left[\text{Using the rules of logs}\right]$$

(b) Converting the given integrand into partial fractions we have

$$\frac{2}{\left(x-1\right)\left(x-2\right)\left(x-3\right)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Since we have linear factors so using the cover up method:

$$A = \frac{2}{\left(1-2\right)\left(1-3\right)} = 1$$

$$B = \frac{2}{\left(2-1\right)\left(2-3\right)} = -2$$

$$C = \frac{2}{\left(3-1\right)\left(3-2\right)} = 1$$

Putting these values into the above and integrating gives

$$\int \frac{2}{\left(x-1\right)\left(x-2\right)\left(x-3\right)}\,\mathrm{d}x = \int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{2}{x-2}\,\mathrm{d}x + \int \frac{1}{x-3}\,\mathrm{d}x$$
$$= \ln\left|x-1\right| - 2\ln\left|x-2\right| + \ln\left|x-3\right| + C$$
$$= \ln\left|\frac{\left(x-1\right)\left(x-3\right)}{\left(x-2\right)^2}\right| + C$$
$$= \ln\left|\frac{x^2-4x+3}{x^2-4x+4}\right| + C$$

(c) In this case we first apply long division to the integrand $\dfrac{x^5-x^3-x^2}{x^2-1}$:

$$\begin{array}{r|l} & x^3 - 1 \\ \hline x^2-1 & x^5 - x^3 - x^2 \\ & -\left(x^5 - x^3\right) \\ \hline & 0 \quad +0 - x^2 \\ & \qquad -\left(-x^2+1\right) \\ \hline & \qquad 0 \quad -1 \end{array}$$

Therefore we have

$$\frac{x^5 - x^3 - x^2}{x^2 - 1} = x^3 - 1 - \frac{1}{x^2 - 1}$$

Integrating the right hand side is much easier:

$$\int \frac{x^5 - x^3 - x^2}{x^2 - 1}\,\mathrm{d}x = \int x^3\,\mathrm{d}x - \int 1\,\mathrm{d}x - \int \frac{1}{x^2 - 1}\,\mathrm{d}x \qquad (*)$$

Converting the last integrand on the right hand side into partial fractions:

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Cover up gives $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ so the last integral is given by

$$\begin{aligned}\int \frac{1}{x^2 - 1}\,\mathrm{d}x &= \frac{1}{2}\left[\int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x\right] \\ &= \frac{1}{2}\left[\ln\left|x-1\right| - \ln\left|x+1\right|\right] = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\end{aligned}$$

Putting this result $\int \frac{1}{x^2 - 1}\,\mathrm{d}x = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$ into (*) and integrating the other terms we have

$$\begin{aligned}\int \frac{x^5 - x^3 - x^2}{x^2 - 1}\,\mathrm{d}x &= \frac{x^4}{4} - x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C \\ &= \frac{1}{4}\left(x^4 - 4x - 2\ln\left|\frac{x-1}{x+1}\right|\right) + C\end{aligned}$$

(d) We are given $\int \frac{4x^3}{\left(x^2+1\right)^2}\,\mathrm{d}x$. Expressing the integrand into partial fractions gives

$$\frac{4x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

Multiplying both sides by $\left(x^2+1\right)^2$ gives

$$4x^3 = (Ax+B)\left(x^2+1\right) + Cx + D \qquad (\dagger)$$

Equating coefficients of x^3 in (†) yields

$$4 = A$$

Equating coefficients of x^2 in (†) yields

$$0 = B$$

Equating coefficients of x in (†) gives

$$0 = A + C = 4 + C \;\Rightarrow\; C = -4$$

Equating coefficients of *constants* in (†) gives

$$0 = B + D = 0 + D \;\Rightarrow\; D = 0$$

Substituting these values into (†):

$$\frac{4x^3}{\left(x^2+1\right)^2} = \frac{4x}{x^2+1} - \frac{4x}{\left(x^2+1\right)^2}$$

Now integrating this

$$\int \frac{4x^3}{\left(x^2+1\right)^2}\,\mathrm{d}x = \int \frac{4x}{x^2+1}\,\mathrm{d}x - \int \frac{4x}{\left(x^2+1\right)^2}\,\mathrm{d}x \qquad (*)$$

Integrating each of these separately.

Therefore

$$\int \frac{4x}{x^2+1}\,\mathrm{d}x = 2\ln\left|x^2+1\right|$$

$$\int \frac{4x}{\left(x^2+1\right)^2}\,\mathrm{d}x \underbrace{=}_{u=x^2+1} \int \frac{4\cancel{x}}{u^2}\frac{\mathrm{d}u}{2\cancel{x}} = 2\int u^{-2}\mathrm{d}u = -\frac{2}{u} = -\frac{2}{x^2+1}$$

Substituting these into (*) and adding the constant of integration gives

$$\int \frac{4x^3}{\left(x^2+1\right)^2}\,\mathrm{d}x = 2\ln\left|x^2+1\right| - \left[-\frac{2}{x^2+1}\right] + C$$

$$= 2\left[\ln\left|x^2+1\right| + \frac{1}{x^2+1}\right] + C$$

(e) In order to find $\int \frac{x^6 - x^2 + 1}{\left(x-1\right)^3}\,\mathrm{d}x$ we apply long division to the integrand.

First we need to expand the denominator of the integrand

$$\left(x-1\right)^3 = x^3 - 3x^2 + 3x - 1$$

Now applying long division:

$$
\begin{array}{r|l}
 & x^3+3x^2+6x+10 \\
\hline
x^3-3x^2+3x-1 & x^6 \qquad\qquad -x^2+1 \\
 & -\left(x^6-3x^5+3x^4-x^3\right) \\
\hline
 & 0+3x^5-3x^4+x^3-x^2+1 \\
 & -\left(3x^5-9x^4+9x^3-3x^2\right) \\
\hline
 & 0+6x^4-8x^3+2x^2\qquad+1 \\
 & -\left(6x^4-18x^3+18x^2-6x\right) \\
\hline
 & 0+10x^3-16x^2+6x+1 \\
 & -\left(10x^3-30x^2+30x-10\right) \\
\hline
 & 0+14x^2-24x+11
\end{array}
$$

Hence we have

$$\frac{x^6-x^2+1}{\left(x-1\right)^3}=x^3+3x^2+6x+10+\frac{14x^2-24x+11}{\left(x-1\right)^3} \qquad (*)$$

Converting the last term on the right hand side into partial fractions gives

$$\frac{14x^2-24x+11}{\left(x-1\right)^3}=\frac{A}{x-1}+\frac{B}{\left(x-1\right)^2}+\frac{C}{\left(x-1\right)^3} \qquad (\dagger)$$

Multiplying both sides by $\left(x-1\right)^3$:

$$14x^2-24x+11=A\left(x-1\right)^2+B\left(x-1\right)+C \qquad (**)$$

Substituting $x=1$ into (**) yields

$$14-24+11=1=C$$

Equating coefficients of x^2 in (**):

$$14=A$$

Equating coefficients of x in (**) gives

$$-24=-2A+B=-2\left(14\right)+B \quad \Rightarrow \quad B=28-24=4$$

Substituting these values into (†) gives

$$\frac{14x^2-24x+11}{\left(x-1\right)^3}=\frac{14}{x-1}+\frac{4}{\left(x-1\right)^2}+\frac{1}{\left(x-1\right)^3}$$

Putting this into (*) yields

$$\frac{x^6-x^2+1}{\left(x-1\right)^3}=x^3+3x^2+6x+10+\frac{14}{x-1}+\frac{4}{\left(x-1\right)^2}+\frac{1}{\left(x-1\right)^3}$$

Integrating this gives

$$\int \frac{x^6 - x^2 + 1}{(x-1)^3}\,\mathrm{d}x = \int \left(x^3 + 3x^2 + 6x + 10\right)\mathrm{d}x + \int \frac{14}{x-1}\,\mathrm{d}x + \int \frac{4}{(x-1)^2}\,\mathrm{d}x + \int \frac{1}{(x-1)^3}\,\mathrm{d}x$$

$$= \frac{x^4}{4} + x^3 + 3x^2 + 10x + 14\ln|x-1| - \frac{4}{x-1} - \frac{1}{2(x-1)^2} + C$$

(f) We are given the integral $\int \frac{3x^2+3}{x^3-3x-2}\,\mathrm{d}x$. Taking out a 3 gives

$$\int \frac{3x^2+3}{x^3-3x-2}\,\mathrm{d}x = 3\int \frac{x^2+1}{x^3-3x-2}\,\mathrm{d}x \qquad (*)$$

Since substituting $x=-1$ into the denominator gives zero so $x+1$ must be a factor of x^3-3x-2. We have

$$x^3 - 3x - 2 = (x+1)\left(x^2 + \alpha x - 2\right)$$

Expanding out the brackets and equating coefficients:

$$x^3 - 3x - 2 = (x+1)\left(x^2 + \alpha x - 2\right) = x^3 + (\alpha+1)x^2 + (\alpha-2)x - 2$$

From this we have

$$(\alpha+1) = 0 \quad \Rightarrow \quad \alpha = -1$$

The cubic term factorizes into

$$x^3 - 3x - 2 = (x+1)\left(x^2 - x - 2\right) = (x+1)(x+1)(x-2) = (x+1)^2(x-2)$$

Converting the integrand on the right hand side of (*) into partial fractions:

$$\frac{x^2+1}{x^3-3x-2} = \frac{x^2+1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Multiplying both sides of the above by $(x+1)^2(x-2)$:

$$x^2 + 1 = A(x+1)(x-2) + B(x-2) + C(x+1)^2 \qquad (**)$$

Substituting $x=-1$ into (**) gives

$$(-1)^2 + 1 = -3B \quad \Rightarrow \quad B = -\frac{2}{3}$$

Substituting $x=2$ into (**) gives

$$2^2 + 1 = C(2+1)^2 \quad \Rightarrow \quad C = \frac{5}{9}$$

Equating constants in (**) yields

$$1 = -2A - 2B + C$$

$$= -2A - 2\left(-\frac{2}{3}\right) + \frac{5}{9} \quad \Rightarrow \quad A = \frac{1}{2}\left[\frac{4}{3} + \frac{5}{9} - 1\right] = \frac{4}{9}$$

Putting these values into (*) gives

$$\begin{aligned}
\int \frac{3x^2+3}{x^3-3x-2}\,\mathrm{d}x &= 3\int \frac{x^2+1}{x^3-3x-2}\,\mathrm{d}x \\
&= 3\int \left(\frac{4}{9\left(x+1\right)} - \frac{2}{3\left(x+1\right)^2} + \frac{5}{9\left(x-2\right)}\right)\mathrm{d}x \\
&= \frac{3}{9}\left[4\int \frac{1}{x+1}\,\mathrm{d}x - 6\int \frac{1}{\left(x+1\right)^2}\,\mathrm{d}x + 5\int \frac{1}{x-2}\,\mathrm{d}x\right] \\
&= \frac{1}{3}\left[4\ln\left|x+1\right| + \frac{6}{x+1} + 5\ln\left|x-2\right|\right] + C
\end{aligned}$$