Complete Solutions to Exercises 17(f)

1. We need to find the Fourier series of the even function:

$$h(t)=\begin{cases}1 & 0<t<\pi \\ 1 & -\pi<t<0\end{cases}$$

Clearly the average value is 1 so $A_0 = 1$.

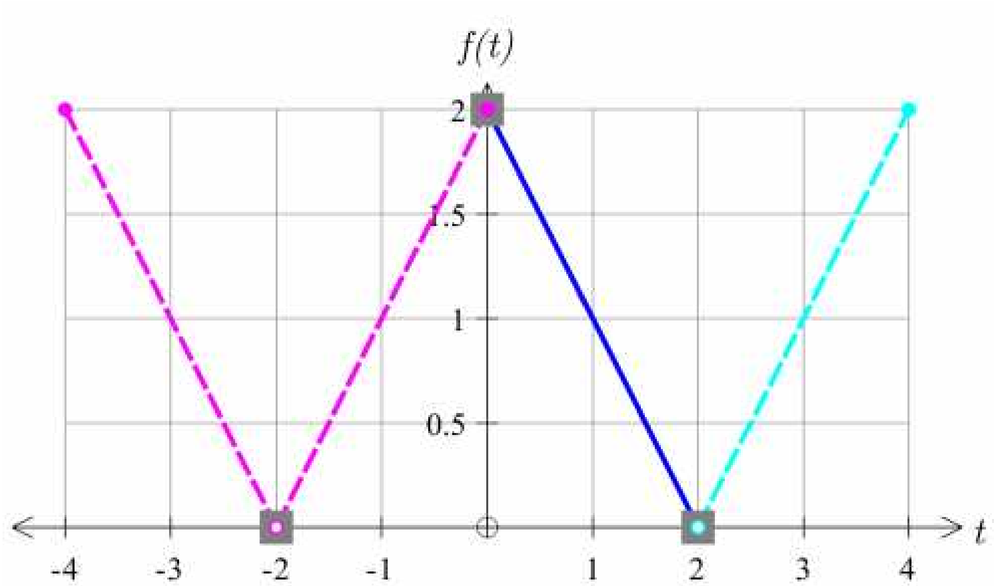
The cosine Fourier coefficients are given by

$$\begin{aligned}A_k &= \frac{2}{\pi}\int_0^{\pi}\left[f(t)\cos(kt)\right]\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi}\left[(1)\cos(kt)\right]\mathrm{d}t && \left[\text{Substituting } f(t)=1\right] \\ &= \frac{2}{k\pi}\left[\sin(kt)\right]_0^{\pi} && \left[\text{Because } \int\cos(kt)\,\mathrm{d}t=\frac{\sin(kt)}{k}\right] \\ &= \frac{2}{k\pi}\underbrace{\left[\sin(k\pi)-0\right]}_{=0} = 0\end{aligned}$$

Hence the Fourier series is given by the constant function:

$$h(t)=1$$

2. The graph of $f(t)$ such that it is even is:



The period $2L=4 \quad \Rightarrow \quad L=2$. We only find the constant term and the cosine coefficients.

The constant term $A_0 = 1$ because this is the average value of the function over a complete period.

We use formula (17.22):

$$A_k = \frac{2}{L}\int_0^L \left[f(t)\cos\left(\frac{k\pi t}{L}\right)\right]dt$$

In our case we have $L = 2$ and $f(t) = 2 - t$ for t between 0 and 2.

Substituting these into the above formula gives

$$A_k = \frac{\not{2}}{\not{2}}\int_0^2 \left[(2-t)\cos\left(\frac{k\pi t}{2}\right)\right] dt$$

We use integration by parts to find this integral. Let

$$u = 2 - t \qquad v' = \cos\left(\frac{k\pi t}{2}\right)$$

$$u' = -1 \qquad v = \int \cos\left(\frac{k\pi t}{2}\right)dt = \frac{1}{k\pi / 2}\sin\left(\frac{k\pi t}{2}\right) = \frac{2}{k\pi}\sin\left(\frac{k\pi t}{2}\right)$$

We have

$$\begin{aligned}
A_k &= \int_0^2 \left[(2-t)\cos\left(\frac{k\pi t}{2}\right)\right] dt \\
&= \frac{2}{k\pi}\left[(2-t)\sin\left(\frac{k\pi t}{2}\right)\right]_0^2 + \frac{2}{k\pi}\int_0^2 \sin\left(\frac{k\pi t}{2}\right) dt \qquad \left[\text{By } \int (uv')\,dt = uv - \int (u'v)\,dt\right] \\
&= \frac{2}{k\pi}\left\{[0] - \frac{1}{k\pi / 2}\left[\cos\left(\frac{k\pi t}{2}\right)\right]_0^2\right\} \qquad \left[\begin{array}{l}\text{First bracket is 0 because} \\ \text{subtituting } t = 2 \text{ and } t = 0 \\ \text{gives 0.}\end{array}\right] \\
&= -\frac{4}{k^2\pi^2}\left[\cos\left(\frac{\not{2}k\pi}{\not{2}}\right) - \cos(0)\right] \\
&= -\frac{4}{k^2\pi^2}\left[\cos(k\pi) - 1\right]
\end{aligned}$$

Now we use the following trigonometric result:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

Substituting the above evaluation of A_k into this yields

$$A_k = -\frac{4}{k^2\pi^2}\left[\cos(k\pi) - 1\right] = \begin{cases} -\dfrac{4}{k^2\pi^2}[1-1] = 0 & \text{if } k = \text{even} \\ -\dfrac{4}{k^2\pi^2}[-1-1] = \dfrac{8}{k^2\pi^2} & \text{if } k = \text{odd} \end{cases}$$

This implies that we only have odd cosine terms.

Substituting $A_0 = 1$, $A_k = \begin{cases} 0 & \text{if } k = \text{even} \\ \dfrac{8}{k^2\pi^2} & \text{if } k = \text{odd} \end{cases}$ and $L = 2$ into the general FS:

(17.20)

$$f(t) = A_0 + A_1 \cos\left(\frac{\pi t}{L}\right) + A_2 \cos\left(\frac{2\pi t}{L}\right) + \cdots + B_1 \sin\left(\frac{\pi t}{L}\right) + B_2 \sin\left(\frac{2\pi t}{L}\right) + \cdots$$

Gives

$$f(t) = 1 + \frac{8}{\pi^2}\cos\left(\frac{\pi t}{2}\right) + \frac{8}{3^2\pi^2}\cos\left(\frac{3\pi t}{2}\right) + \frac{8}{5^2\pi^2}\cos\left(\frac{5\pi t}{2}\right) + \cdots$$
$$= 1 + \frac{8}{\pi^2}\left[\cos\left(\frac{\pi t}{2}\right) + \frac{\cos(3\pi t/2)}{3^2} + \frac{\cos(5\pi t/2)}{5^2} + \cdots\right]$$

(ii) Substituting $t = 0$ into this

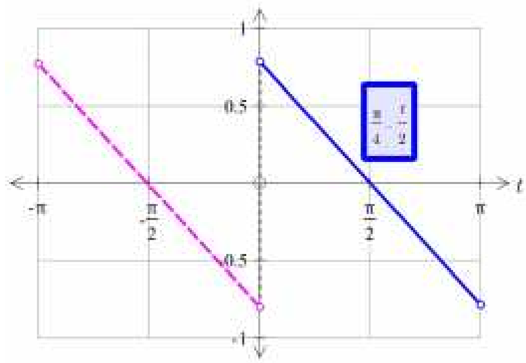
$$f(t) = 1 + \frac{8}{\pi^2}\left[\cos\left(\frac{\pi t}{2}\right) + \frac{\cos(3\pi t/2)}{3^2} + \frac{\cos(5\pi t/2)}{5^2} + \cdots\right]$$

Gives

$$f(0) = 1 + \frac{8}{\pi^2}\left[\cos(0) + \frac{\cos(0)}{3^2} + \frac{\cos(0)}{5^2} + \frac{\cos(0)}{7^2} + \cdots\right]$$
$$2 = 1 + \frac{8}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$
$$1 = \frac{8}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

This is our required result.

3. (i) The graph of odd $f(t)$ is



The period is given as 2π . We use the following B_k formula to find the sine coefficients:

(17.13)	$B_k = \dfrac{2}{\pi}\int_0^{\pi}\left[f(t)\sin(kt)\right]\mathrm{d}t$

We are given that $f(t) = \dfrac{\pi}{4} - \dfrac{t}{2}$ between 0 and π . Therefore

$$B_k = \frac{2}{\pi}\int_0^{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right)\sin(kt)\right]\mathrm{d}t \qquad (*)$$

We use integration by parts to find the right hand integral in (*). Let

$$u = \frac{\pi}{4} - \frac{t}{2} \qquad v' = \sin(kt)$$

$$u' = -\frac{1}{2} \qquad v = \int \sin(kt)\,\mathrm{d}t = -\frac{1}{k}\cos(kt)$$

Putting these into the integral gives

$$\begin{aligned}
\int_0^{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right)\sin(kt)\right]\mathrm{d}t &= \left[uv\right]_0^{\pi} - \int_0^{\pi}(u'v)\,\mathrm{d}t \\
&= -\frac{1}{k}\left\{\left[\left(\frac{\pi}{4} - \frac{t}{2}\right)\cos(kt)\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos(kt)\,\mathrm{d}t\right\} \\
&= -\frac{1}{k}\left\{\left[\left(\frac{\pi}{4} - \frac{\pi}{2}\right)\cos(k\pi) - \frac{\pi}{4}\right] + \frac{1}{2k}\left[\sin(kt)\right]_0^{\pi}\right\} \\
&= -\frac{1}{k}\left\{\left[-\frac{\pi}{4}\cos(k\pi) - \frac{\pi}{4}\right] + \frac{1}{2k}\underbrace{\left[\sin(k\pi) - \sin(0)\right]}_{=0}\right\} \\
&= \frac{\pi}{4k}\left[\cos(k\pi) + 1\right]
\end{aligned}$$

Putting this $\int_0^{\pi}\left[\left(\dfrac{\pi}{4} - \dfrac{t}{2}\right)\sin(kt)\right]\mathrm{d}t = \dfrac{\pi}{4k}\left[\cos(k\pi) + 1\right]$ into (*) gives

$$\begin{aligned}
B_k &= \frac{2}{\pi}\int_0^{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right)\sin(kt)\right]\mathrm{d}t \\
&= \frac{2}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{4k}\left[\cos(k\pi) + 1\right]\right) = \frac{1}{2k}\left[\cos(k\pi) + 1\right] \qquad (**)
\end{aligned}$$

Recall that

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

Substituting this into (**) yields

$$B_k = \frac{1}{2k}\left[\cos(k\pi) - 1\right] = \begin{cases} \dfrac{1}{2k}\left[1+1\right] = \dfrac{1}{k} & \text{if} \quad k = \text{even} \\ -\dfrac{1}{2k}\left[-1+1\right] = 0 & \text{if} \quad k = \text{odd} \end{cases}$$

We only have even sine terms in the Fourier series and this is given by

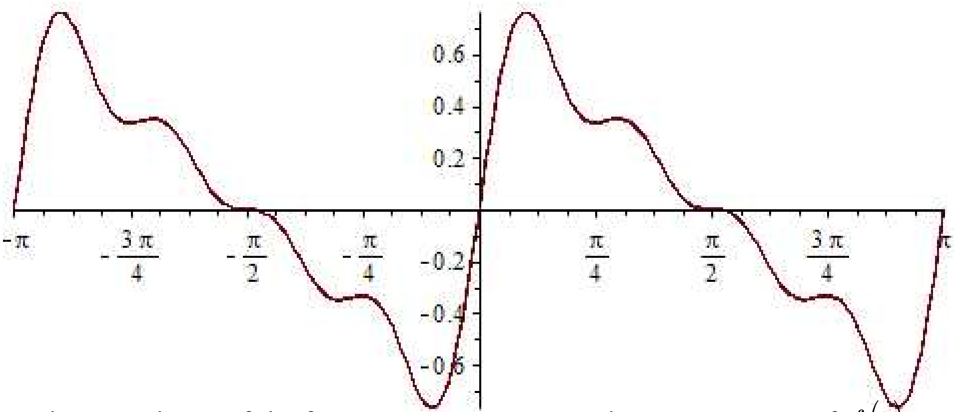
$$f(t) = \frac{1}{2}\sin(2t) + \frac{1}{4}\sin(4t) + \frac{1}{6}\sin(6t) + \frac{1}{8}\sin(8t) + \cdots$$

We can see this graphically in Maple:

$$> f := t \to \frac{\sin(2t)}{2} + \frac{\sin(4t)}{4} + \frac{\sin(6t)}{6} + \frac{\sin(8t)}{8}$$

$$t \to \frac{1}{2}\sin(2t) + \frac{1}{4}\sin(4t) + \frac{1}{6}\sin(6t) + \frac{1}{8}\sin(8t)$$

$> plot(f, -\pi..\pi)$



The partial sum of the first 4 non-zero terms in the Fourier series of $f(t)$.

(ii) Substituting $t = \dfrac{\pi}{4}$ into the above Fourier series

$$f(t) = \frac{1}{2}\sin(2t) + \frac{1}{4}\sin(4t) + \frac{1}{6}\sin(6t) + \frac{1}{8}\sin(8t) + \cdots$$

Gives

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \frac{1}{2}\sin\left(2\frac{\pi}{4}\right) + \frac{1}{4}\sin\left(4\frac{\pi}{4}\right) + \frac{1}{6}\sin\left(6\frac{\pi}{4}\right) + \frac{1}{8}\sin\left(8\frac{\pi}{4}\right) + \frac{1}{10}\sin\left(10\frac{\pi}{4}\right) + \cdots \\ &= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin(\pi) + \frac{1}{6}\sin\left(3\frac{\pi}{2}\right) + \frac{1}{8}\sin(2\pi) + \frac{1}{10}\sin\left(5\frac{\pi}{2}\right) + \cdots \\ &= \frac{1}{2}(1) + \frac{1}{4}(0) + \frac{1}{6}(-1) + \frac{1}{8}(0) + \frac{1}{10}(1) + \cdots \\ &= \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \cdots \end{aligned}$$

What is $f\left(\dfrac{\pi}{4}\right)$ *equal to?*

We are given $f(t)=\dfrac{\pi}{4}-\dfrac{t}{2}$ if $0<t<\pi$. Since $0<\dfrac{\pi}{4}<\pi$ so

$$f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}-\frac{\pi/4}{2}=\frac{\pi}{4}-\frac{\pi}{8}=\frac{\pi}{8}$$

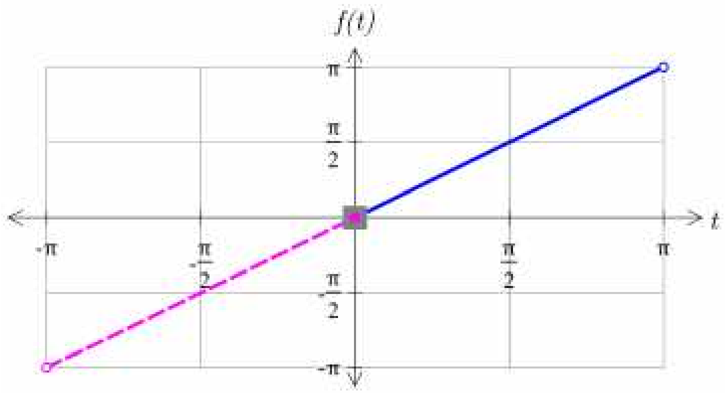
Hence we have

$$f\left(\frac{\pi}{4}\right)=\frac{\pi}{8}=\frac{1}{2}-\frac{1}{6}+\frac{1}{10}-\frac{1}{14}+\cdots$$

4. (a) We need to sketch an odd function extension to

$$f(t)=t \qquad \text{if } 0\le t<\pi$$

This is $f(t)=\begin{cases} t & \text{if } \quad 0\le t<\pi \\ t & \text{if } \ -\pi<t\le 0 \end{cases}$. The graph is



We have a period of 2π and it is an odd function so we use the formula

$$B_k=\frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t \qquad (*)$$

We use integration by parts to evaluate the integral on the right hand side. Let

$$u=t \qquad v'=\sin(kt)$$
$$u'=1 \qquad v=\int\sin(kt)\,\mathrm{d}t=-\frac{1}{k}\cos(kt)$$

We have

$$\int_0^\pi \left[t\sin\left(kt\right)\right]\mathrm{d}t = uv - \int\left(u'v\right)\mathrm{d}t$$

$$= -\frac{1}{k}\left\{\left[t\cos\left(kt\right)\right]_0^\pi - \int_0^\pi \cos\left(kt\right)\mathrm{d}t\right\}$$

$$= -\frac{1}{k}\left\{\left[\pi\cos\left(k\pi\right) - 0\right] - \frac{1}{k}\left[\sin\left(kt\right)\right]_0^\pi\right\}$$

$$= -\frac{1}{k}\left\{\pi\cos\left(k\pi\right) - \frac{1}{k}\underbrace{\left[\sin\left(k\pi\right) - \sin\left(0\right)\right]}_{=0}\right\}$$

$$= -\frac{\pi}{k}\cos\left(k\pi\right)$$

Substituting this $\int_0^\pi \left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi}{k}\cos\left(k\pi\right)$ into (*) yields

$$B_k = \frac{2}{\pi}\int_0^\pi f\left(t\right)\sin\left(kt\right)\mathrm{d}t$$

$$= -\frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{k}\cos\left(k\pi\right)\right] = -\frac{2}{k}\cos\left(k\pi\right)$$

Using our trigonometric result

$$\cos\left(k\pi\right) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

We have

$$B_k = -\frac{2}{k}\cos\left(k\pi\right) = \begin{cases} -2/k & \text{if } k = \text{even} \\ 2/k & \text{if } k = \text{odd} \end{cases}$$

Substituting this into the generic Fourier series with only sine terms gives

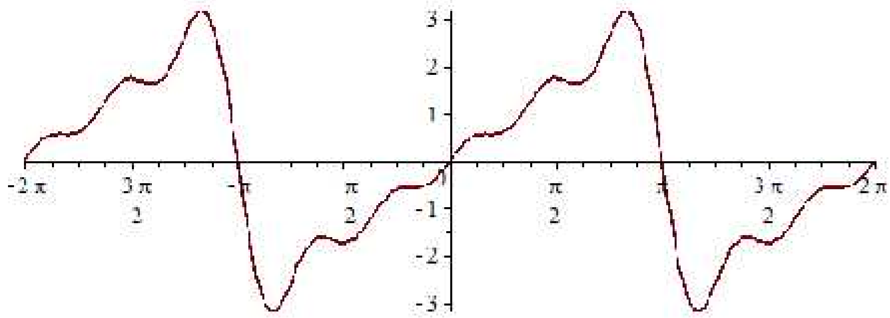
$$f\left(t\right) = 2\left[\sin\left(t\right) - \frac{\sin\left(2t\right)}{2} + \frac{\sin\left(3t\right)}{3} - \frac{\sin\left(4t\right)}{4} + \frac{\sin\left(5t\right)}{5} - \cdots\right]$$

We can see the graphical output in Maple:

> $f := t \rightarrow 2\left(\sin(t) - \frac{\sin(2\,t)}{2} + \frac{\sin(3\,t)}{3} - \frac{\sin(4\,t)}{4} + \frac{\sin(5\,t)}{5}\right)$

$$t \rightarrow 2\sin(t) - \sin(2\,t) + \frac{2}{3}\sin(3\,t) - \frac{1}{2}\sin(4\,t) + \frac{2}{5}\sin(5\,t)$$

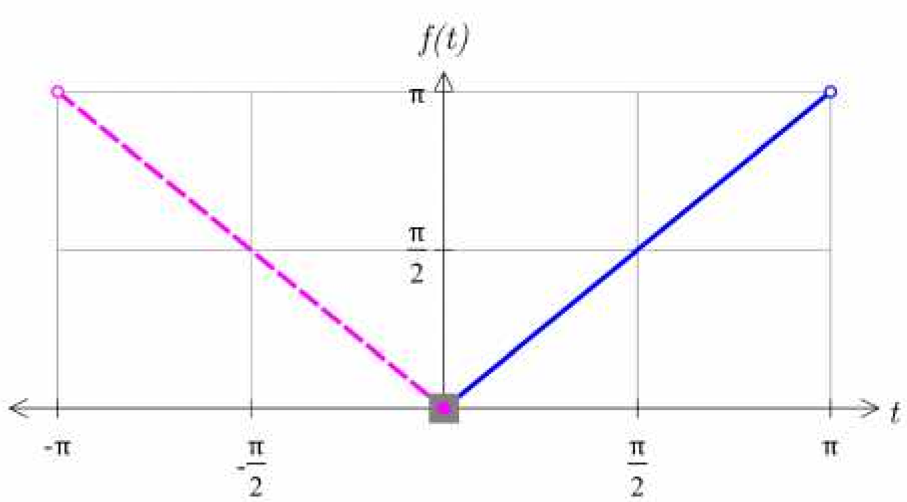
> $plot(f, -2\,\pi..2\,\pi)$



(b) This time we need to sketch an even function extension to

$$f(t) = t \qquad \text{if } \ 0 \le t < \pi$$

This is $f(t) = \begin{cases} t & \text{if} \quad 0 \le t < \pi \\ -t & \text{if} \quad -\pi < t \le 0 \end{cases}$ and the graph is:



We have a period of 2π and it is an even function. The average value of this function is $\frac{\pi}{2}$ so $A_0 = \frac{\pi}{2}$. We use the following formula to evaluate the A_k's:

$$A_k = \frac{2}{\pi}\int_0^\pi f(t)\cos(kt)\,\mathrm{d}t \qquad (**)$$

We use integration by parts to evaluate the integral on the right hand side. Let

$$u = t \qquad v' = \cos(kt)$$

$$u' = 1 \qquad v = \int \cos(kt)\,\mathrm{d}t = \frac{1}{k}\sin(kt)$$

We have

$$\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = uv - \int (u'v)\,\mathrm{d}t$$
$$= \frac{1}{k}\left\{\left[t\sin(kt)\right]_0^{\pi} - \int_0^{\pi}\sin(kt)\,\mathrm{d}t\right\}$$
$$= \frac{1}{k}\left\{\left[\pi\underbrace{\sin(k\pi)}_{=0} - 0\right] + \frac{1}{k}\left[\cos(kt)\right]_0^{\pi}\right\}$$
$$= \frac{1}{k}\left\{0 + \frac{1}{k}\left[\cos(k\pi) - \cos(0)\right]\right\}$$
$$= \frac{1}{k^2}\left[\cos(k\pi) - 1\right]$$

Substituting this $\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = \frac{1}{k^2}\left[\cos(k\pi)-1\right]$ into (**) yields

$$A_k = \frac{2}{\pi}\left(\frac{1}{k^2}\left[\cos(k\pi)-1\right]\right) = \frac{2}{k^2\pi}\left[\cos(k\pi)-1\right]$$

Using our trigonometric result

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

We have

$$A_k = \frac{2}{k^2\pi}\left[\cos(k\pi)-1\right] = \begin{cases} 0 & \text{if } k = \text{even} \\ \frac{2}{k^2\pi}\left[-2\right] = -\frac{4}{k^2\pi} & \text{if } k = \text{odd} \end{cases}$$

Substituting this into the generic Fourier series with only odd cosine terms and $A_0 = \frac{\pi}{2}$:

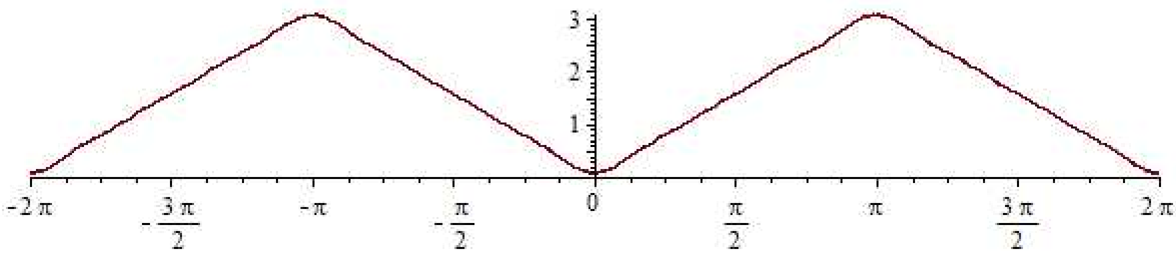
$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \frac{\cos(7t)}{7^2} + \cdots\right]$$

We can see the graphical output in Maple:

$> g := t \to \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \frac{\cos(7t)}{49}\right)$

$$t \to \frac{1}{2}\pi - \frac{4\left(\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \frac{1}{49}\cos(7t)\right)}{\pi}$$

$> plot(g, -2\pi..2\pi)$

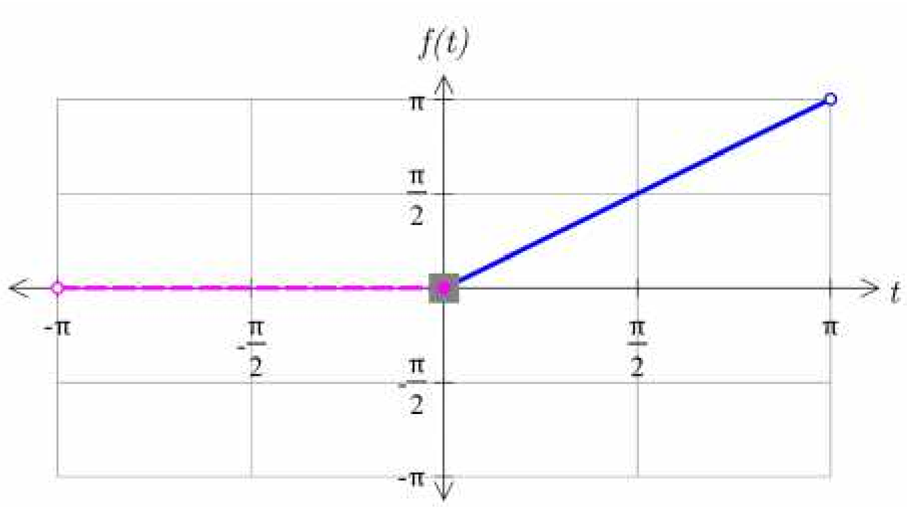


(c) This time we need to extend the interval so that the Fourier series contains both sine and cosine terms:

$$f(t)=\begin{cases} t & \text{if} \quad 0 \le t < \pi \\ 0 & \text{if} \quad -\pi < t \le 0 \end{cases}$$

There are an infinite number of choices for this extension and we have chosen $f(t)=0$ between $-\pi$ and 0 because it will cut down the amount of arithmetic needed to evaluate the Fourier coefficients.

The graph of the function is



We have a period of 2π. *What is the average value of this function?*

The area of the triangle on the above right is $\frac{\pi^2}{2}$. Therefore

$$A_0 = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{4}$$

The sine coefficients are given by

$$B_k = \frac{1}{\pi}\int_0^{\pi} t\sin(kt)\,\mathrm{d}t$$

This means they are half the coefficients evaluated in part (a) because in that case we had $B_k = \frac{2}{\pi}\int_0^{\pi} t\sin(kt)\,dt$. Therefore

$$B_k = \begin{cases} -1/k & \text{if } k = \text{even} \\ 1/k & \text{if } k = \text{odd} \end{cases}$$

Similarly from part (b) we have

$$A_k = \begin{cases} 0 & \text{if } k = \text{even} \\ -\dfrac{2}{k^2\pi} & \text{if } k = \text{odd} \end{cases}$$

Putting all these values into the general Fourier series formula:

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$$

Gives

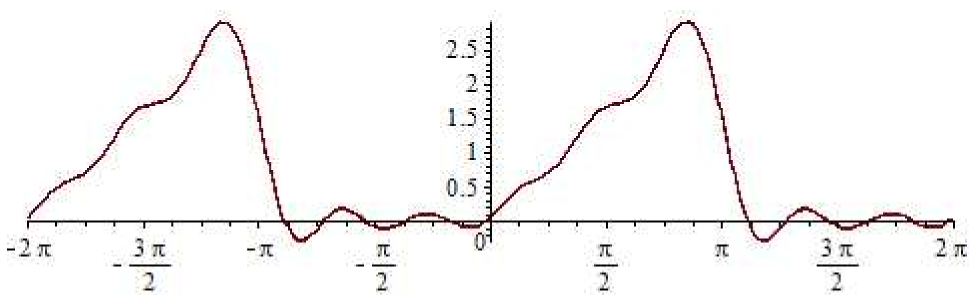
$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right] + \left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \cdots\right]$$

The graphical Maple output is

$$> h := t \to \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \frac{\cos(7t)}{49}\right) + \sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \frac{\sin(5t)}{5}$$

$$t \to \frac{1}{4}\pi - \frac{2\left(\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \frac{1}{49}\cos(7t)\right)}{\pi} + \sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) + \frac{1}{5}\sin(5t)$$

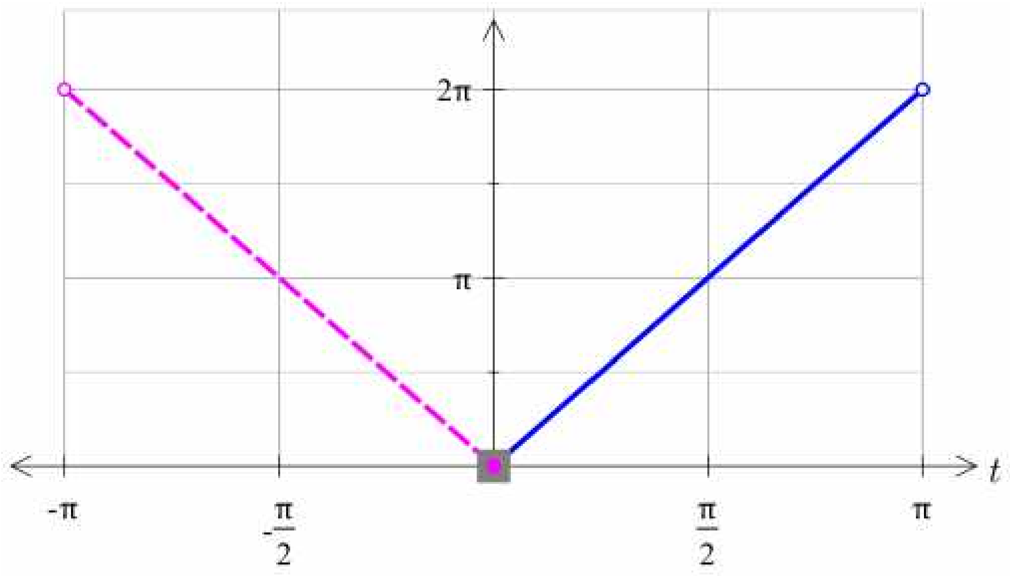
$> plot(h, -2\pi..2\pi)$



5. We need to find the cosine Fourier series of

$$f(t) = 2t \qquad \text{when} \quad 0 \le t < \pi$$

We have to extend this function so that it is an even function. We have



Let $g(t)$ be this function so

$$g(t) = \begin{cases} 2t & \text{when} \quad 0 \le t < \pi \\ -2t & \text{when} \quad -\pi < t \le 0 \end{cases}$$

Note that this is 2 times the function $f(t)$ of the previous question part (b) because

$$f(t) = \begin{cases} t & \text{if} \quad 0 \le t < \pi \\ -t & \text{if} \quad -\pi < t \le 0 \end{cases}$$

Multiplying this by 2 gives

$$g(t) = 2f(t) = \begin{cases} 2t & \text{when} \quad 0 \le t < \pi \\ -2t & \text{when} \quad -\pi < t \le 0 \end{cases}$$

By using the linearity property of section B:

> The Fourier series of $cf(t)$ where c is a constant is
>
> $$cf(t) = cA_0 + cA_1\cos(t) + cA_2\cos(2t) + \cdots + cB_1\sin(t) + cB_2\sin(2t) + \cdots$$

The Fourier series found in question 4(b) was

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \frac{\cos(7t)}{7^2} + \cdots\right]$$

Therefore $g(t)$ is double of this

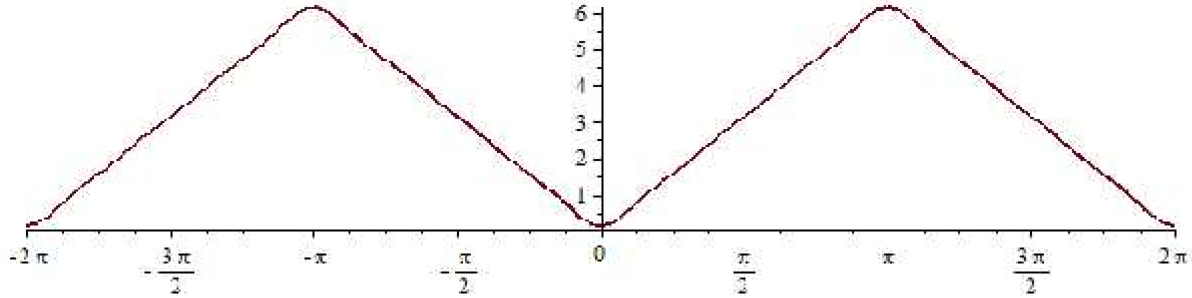
$$g(t) = 2f(t) = \pi - \frac{8}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \frac{\cos(7t)}{7^2} + \cdots\right]$$

The graphical output can be seen by using Maple:

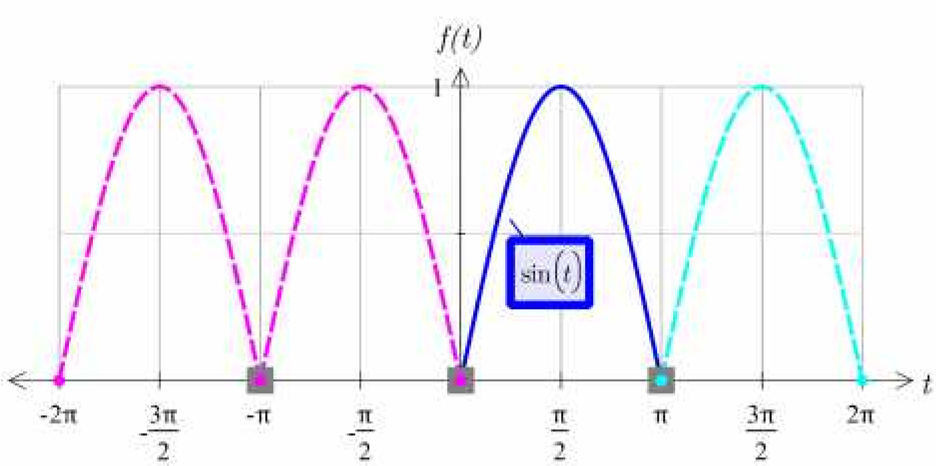
> $g := t \to \pi - \dfrac{8}{\pi}\left(\cos(t) + \dfrac{\cos(3\,t)}{9} + \dfrac{\cos(5\,t)}{25} + \dfrac{\cos(7\,t)}{49}\right)$

$$t \to \pi - \frac{8\left(\cos(t) + \frac{1}{9}\cos(3\,t) + \frac{1}{25}\cos(5\,t) + \frac{1}{49}\cos(7\,t)\right)}{\pi}$$

> $plot(g, -2\,\pi..2\,\pi)$



6. (i) We are given that



Clearly this is an even function as it is symmetrical about the vertical axis.

We only have to find the constant term and the cosine coefficients.

The constant term is given by

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(t)\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} \sin(t)\,\mathrm{d}t$$

$$= -\frac{1}{\pi}\big[\cos(t)\big]_0^{\pi} = -\frac{1}{\pi}\Big[\underbrace{\cos(\pi)}_{=-1} - \underbrace{\cos(0)}_{=1}\Big] = -\frac{1}{\pi}\big[-1-1\big] = \frac{2}{\pi}$$

We use the following formula to find the cosine coefficients:

$$\text{(17.15)} \qquad A_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

We have

$$A_k = \frac{2}{\pi}\int_0^\pi \left[f(t)\cos(kt)\right]\mathrm{d}t = \frac{2}{\pi}\int_0^\pi \left[\sin(t)\cos(kt)\right]\mathrm{d}t \qquad (*)$$

How do we find this integral $\int_0^\pi \left[\sin(t)\cos(kt)\right]\mathrm{d}t$ *?*

By using the following trigonometric identity:

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Applying this to the above integrand we have

$$\sin(t)\cos(kt) = \frac{1}{2}\left[\sin\left((k+1)t\right) + \left(\sin(1-k)t\right)\right]$$

Now we can evaluate the above integral

$$\begin{aligned}
\int_0^\pi \left[\sin(t)\cos(kt)\right]\mathrm{d}t &= \frac{1}{2}\left\{\int_0^\pi \sin\left((k+1)t\right)\mathrm{d}t + \int_0^\pi \left(\sin(1-k)t\right)\mathrm{d}t\right\} \\
&= \frac{1}{2}\left\{\left[-\frac{\cos\left((k+1)t\right)}{k+1}\right]_0^\pi + \left[-\frac{\cos\left((1-k)t\right)}{1-k}\right]_0^\pi\right\} \\
&\underbrace{=}_{\substack{\text{Taking out} \\ \text{the minus sign} \\ \text{from brackets}}} \frac{1}{2}\left\{\frac{1}{k+1}\left[\cos\left((k+1)\pi\right) - \cos(0)\right] + \frac{1}{1-k}\left[\cos\left((1-k)\pi\right) - \cos(0)\right]\right\} \\
&= -\frac{1}{2}\left\{\frac{1}{k+1}\left[\cos\left((k+1)\pi\right) - 1\right] + \frac{1}{1-k}\left[\cos\left((1-k)\pi\right) - 1\right]\right\}
\end{aligned}$$

If k is even then both $k+1$ and $1-k$ are odd and so

$$\cos\left((k+1)\pi\right) = \cos\left((1-k)\pi\right) = -1$$

Putting this into the above evaluation yields

$$\begin{aligned}
\int_0^\pi \left[\sin(t)\cos(kt)\right]\mathrm{d}t &= -\frac{1}{2}\left\{\frac{1}{k+1}\left[\cos\left((k+1)\pi\right) - 1\right] + \frac{1}{1-k}\left[\cos\left((1-k)\pi\right) - 1\right]\right\} \\
&= -\frac{1}{2}\left\{\frac{1}{k+1}\left[-1-1\right] + \frac{1}{1-k}\left[-1-1\right]\right\} \\
&= -\frac{1}{2}\left\{\frac{-2}{k+1} + \frac{-2}{1-k}\right\} \\
&= \frac{1}{k+1} + \frac{1}{1-k} \\
&= \frac{1-k+k+1}{(k+1)(1-k)} = -\frac{2}{(k+1)(k-1)} = -\frac{2}{k^2-1}
\end{aligned}$$

If k is odd then both $k+1$ and $1-k$ are even and so

$$\cos\left((k+1)\pi\right) = \cos\left((1-k)\pi\right) = 1$$

Substituting this into the above calculation gives

$$\int_0^{\pi}\left[\sin(t)\cos(kt)\right]\mathrm{d}t = -\frac{1}{2}\left\{\frac{1}{k+1}\left[\underbrace{\cos\left((k+1)\pi\right)}_{=1}-1\right]+\frac{1}{1-k}\left[\underbrace{\cos\left((1-k)\pi\right)}_{=1}-1\right]\right\}$$
$$= 0$$

We have

$$\int_0^{\pi}\left[\sin(t)\cos(kt)\right]\mathrm{d}t = \begin{cases} -\dfrac{2}{k^2-1} & \text{if } k = \text{even} \\ 0 & \text{if } k = \text{odd}\end{cases}$$

Putting this into (*) gives

$$A_k = \frac{2}{\pi}\int_0^{\pi}\left[\sin(t)\cos(kt)\right]\mathrm{d}t = \begin{cases} -\dfrac{4}{\pi\left(k^2-1\right)} & \text{if } k = \text{even} \\ 0 & \text{if } k = \text{odd}\end{cases}$$

Evaluating the first few even coefficients (the odd ones are zero):

$$A_2 = -\frac{4}{\pi\left(2^2-1\right)},\quad A_4 = -\frac{4}{\pi\left(4^2-1\right)},\quad A_6 = -\frac{4}{\pi\left(6^2-1\right)},\ \ldots$$

Putting these and $A_0 = \dfrac{2}{\pi}$ into the general Fourier series

$$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$$

Yields

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi\left(2^2-1\right)}\cos(2t) - \frac{4}{\pi\left(4^2-1\right)}\cos(4t) - \frac{4}{\pi\left(6^2-1\right)}\cos(6t) + \cdots$$
$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2t)}{2^2-1} + \frac{\cos(4t)}{4^2-1} + \frac{\cos(6t)}{6^2-1} + \cdots\right]$$

(ii) Putting $t = 0$ into the derived Fourier series of part (i) gives

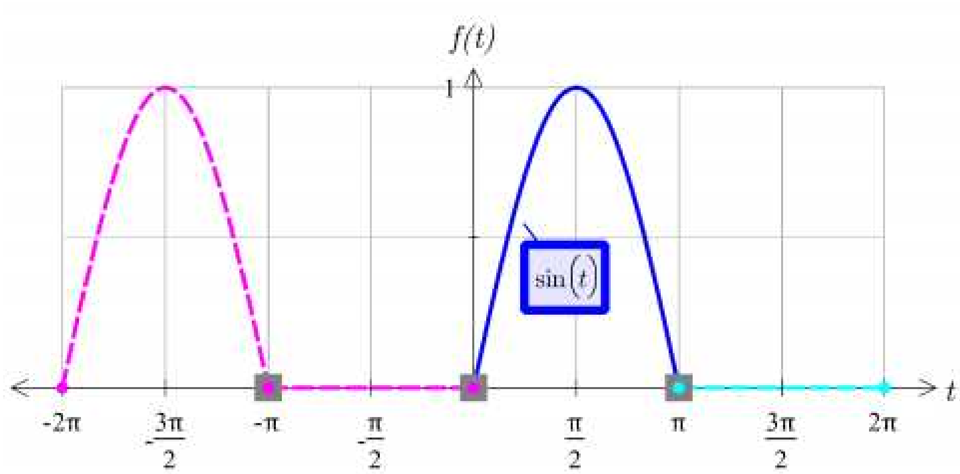
$$f(0) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(0)}{2^2-1} + \frac{\cos(0)}{4^2-1} + \frac{\cos(0)}{6^2-1} + \cdots\right]$$
$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \cdots\right]$$
$$\frac{4}{\pi}\left[\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \cdots\right] = \frac{2}{\pi}$$
$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \cdots = \frac{2}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{4}\right) = \frac{1}{2}$$

We have derived our required result.

(iii) The sine series of $f(t) = \sin(t),\ 0 \le t \le \pi$ is just going to be

$$f(t) = \sin(t),\ 0 \le t \le 2\pi$$

7. (i) We extend the given function so that the graph is given by:



This is denoted by $f(t)=\begin{cases}\sin(t) & \text{if } 0\le t\le \pi \\ 0 & \text{if } \pi < t \le 2\pi\end{cases}$

(ii) We need to find the constant term A_0, the cosine coefficients, A_k, and the sine coefficients B_k.

<u>The constant term</u>:

We have

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} \sin(t)\,\mathrm{d}t$$
$$= -\frac{1}{2\pi}\left[\cos(t)\right]_0^{\pi} = -\frac{1}{2\pi}\left[\underbrace{\cos(\pi)}_{=-1} - 1\right] = \frac{1}{\pi}$$

<u>The cosine coefficients</u>:

We have

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} \sin(t)\cos(kt)\,\mathrm{d}t$$

We evaluated the following integral in the previous question 6 part (i):

$$A_k = \frac{2}{\pi}\int_0^{\pi}\left[\sin(t)\cos(kt)\right]\mathrm{d}t = \begin{cases} -\dfrac{4}{\pi(k^2-1)} & \text{if } k = \text{even} \\ 0 & \text{if } k = \text{odd}\end{cases}$$

Therefore in our case we have half of this because of the $1/\pi$ rather than $2/\pi$ which is

$$A_k = \begin{cases} -\dfrac{2}{\pi(k^2-1)} & \text{if } k = \text{even} \\ 0 & \text{if } k = \text{odd}\end{cases}$$

<u>The sine coefficients</u>:

We have

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t = \frac{1}{\pi}\int_{0}^{\pi} \sin(t)\sin(kt)\,\mathrm{d}t$$

We use the given helpful result:

$$\int_0^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \end{cases}$$

Using this in evaluating B_k gives

$$B_k = \frac{1}{\pi}\int_0^{\pi} \sin(t)\sin(kt)\,\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq 1 \\ 1/2 & \text{if } k = 1 \end{cases}$$

Hence there is only one non-zero B_k which is $B_1 = \frac{1}{2}$.

Substituting these $A_0 = \frac{1}{\pi}$, $A_k = -\frac{2}{\pi(k^2-1)}$ if k is even and $B_1 = \frac{1}{2}$ into the generic Fourier series

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$

Gives

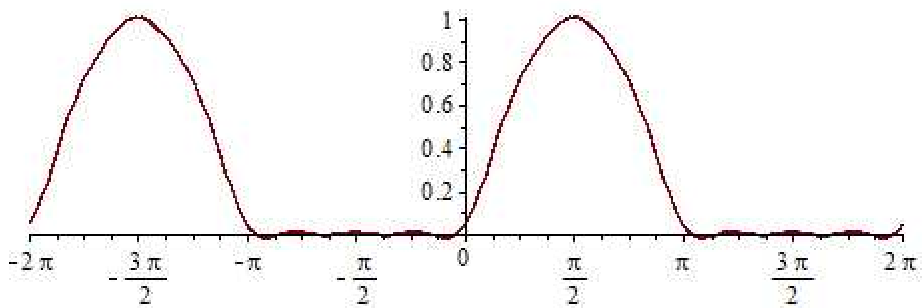
$$f(t) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos(2t)}{2^2-1} + \frac{\cos(4t)}{4^2-1} + \frac{\cos(6t)}{6^2-1} + \cdots\right] + \frac{1}{2}\sin(t)$$

We can see the graphical output of these terms in Fourier series by using Maple:

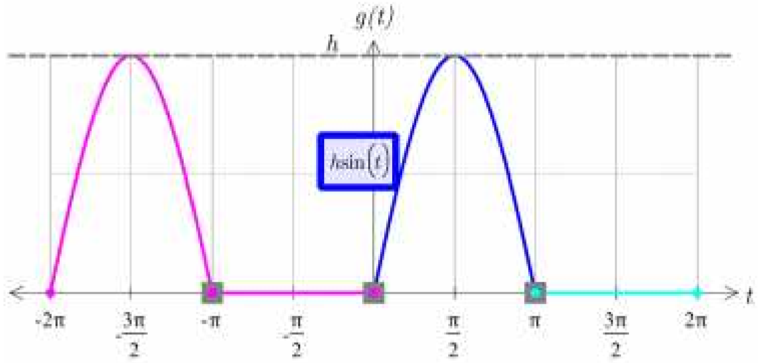
> $f := t \to \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos(2t)}{3} + \frac{\cos(4t)}{15} + \frac{\cos(6t)}{35}\right) + \frac{1}{2}\sin(t)$

$$t \to \frac{1}{\pi} - \frac{2\left(\frac{1}{3}\cos(2t) + \frac{1}{15}\cos(4t) + \frac{1}{35}\cos(6t)\right)}{\pi} + \frac{1}{2}\sin(t)$$

> $plot(f, -2\pi..2\pi)$



8. The given waveform is identical to the one in the solution to question 7 apart from the amplitude which is h:



We have $g(t)=\begin{cases} h\sin(t) & \text{if } 0\le t\le \pi \\ 0 & \text{if } \pi < t \le 2\pi \end{cases}$. From solution to the previous question we have $f(t)=\begin{cases} \sin(t) & \text{if } 0\le t\le \pi \\ 0 & \text{if } \pi < t \le 2\pi \end{cases}$.

Therefore $g(t)=hf(t)$. Applying the linearity property of the Fourier series given in section B we have

$$g(t)=hf(t)=\frac{h}{\pi}-\frac{2h}{\pi}\left[\frac{\cos(2t)}{2^2-1}+\frac{\cos(4t)}{4^2-1}+\frac{\cos(6t)}{6^2-1}+\cdots\right]+\frac{h}{2}\sin(t)$$

This is our required result.