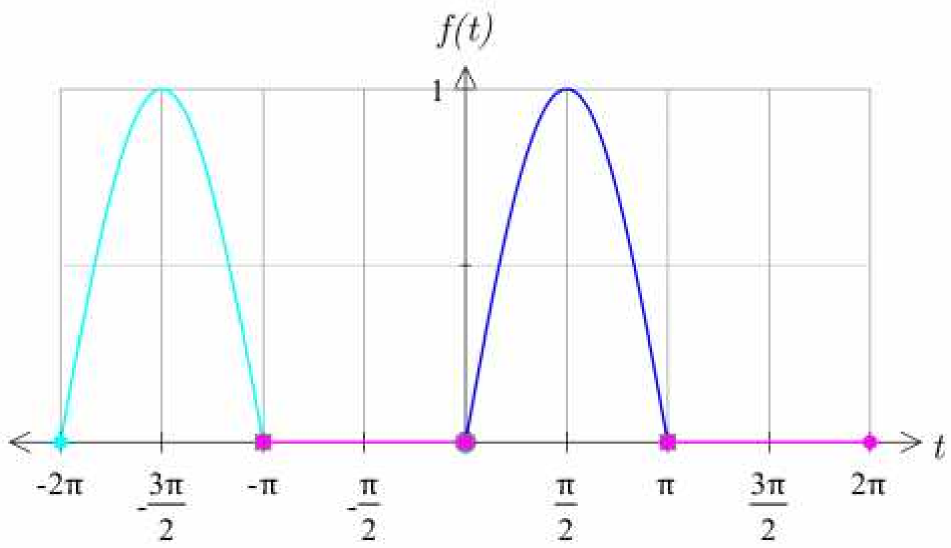
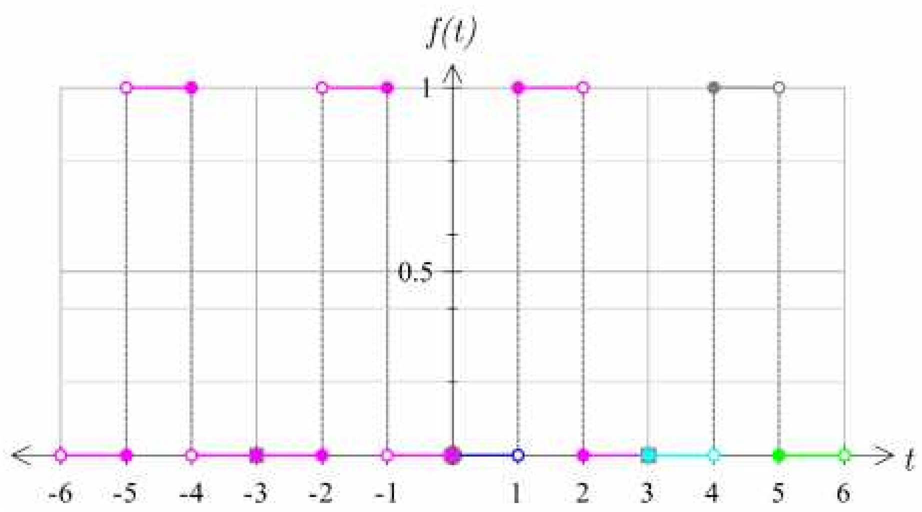
Complete Solutions to Exercises 17(e)

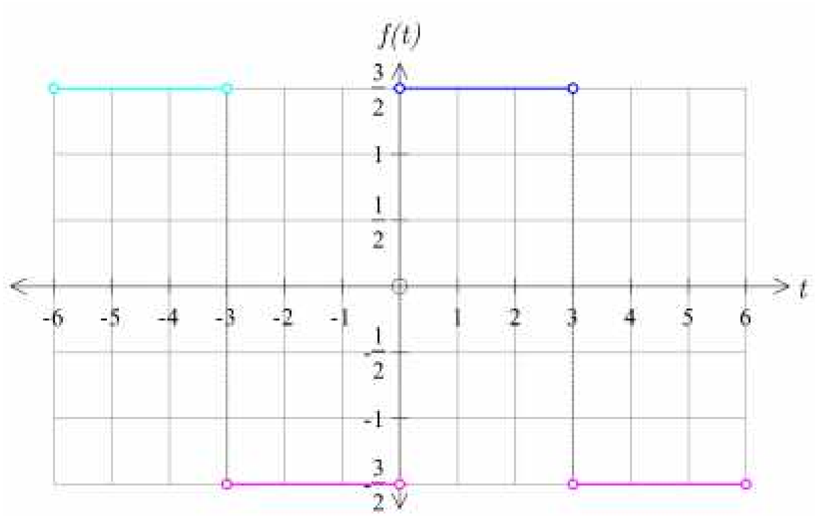
1. (a) We are asked to graph $f(t)=\begin{cases}\sin(t) & 0\le t\le \pi \\ 0 & \pi\le t\le 2\pi\end{cases}$. It is given by:



(b) We need to sketch $f(t)=\begin{cases}0 & 0\le t<1 \\ 1 & 1\le t<2 \\ 0 & 2\le t<3\end{cases}$ with period 3:



2. The given waveform is



Clearly this has a period of 6. We have $2L = 6$ which implies $L = 3$. *What type of waveform do we have?*

Odd. Therefore there are *no* constant or cosine terms in the Fourier series of $f(t)$. The sine coefficients B_k are given by

(17.23) $$B_k = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Substituting $L = 3$ and $f(t) = \frac{3}{2}$, because the above graph shows a value of $\frac{3}{2}$ between 0 and 3, into formula (17.23) gives

$$\begin{aligned}
B_k &= \frac{2}{3}\int_0^3 \frac{3}{2}\sin\left(\frac{k\pi t}{3}\right)\mathrm{d}t \\
&= \int_0^3 \sin\left(\frac{k\pi t}{3}\right)\mathrm{d}t \\
&= -\left[\frac{\cos(k\pi t/3)}{k\pi/3}\right]_0^3 \qquad \left[\text{By } \int \sin(mt)\,\mathrm{d}t = -\frac{\cos(mt)}{m}\right] \\
&= -\frac{3}{k\pi}\left[\cos\left(\frac{3k\pi}{3}\right) - \cos(0)\right] \\
&= -\frac{3}{k\pi}\left[\cos(k\pi) - 1\right]
\end{aligned}$$

We have $B_k = -\frac{3}{k\pi}\left[\cos(k\pi) - 1\right]$. Recall that

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

Clearly if k is even we have $1 - 1 = 0$ in the square brackets of B_k so $B_k = 0$.

If k is odd we have

$$B_k = -\frac{3}{k\pi}\left[\cos\left(k\pi\right)-1\right] = -\frac{3}{k\pi}\left[-1-1\right] = \frac{6}{k\pi}$$

We only have odd sine terms in the Fourier series of $f\left(t\right)$:

$$f\left(t\right) = B_1 \sin\left(\frac{\pi t}{L}\right) + B_3 \sin\left(\frac{3\pi t}{L}\right) + B_5 \sin\left(\frac{5\pi t}{L}\right) + B_7 \sin\left(\frac{7\pi t}{L}\right) + \cdots$$

Therefore our Fourier series is given by this with $L = 3$ and $B_k = \dfrac{6}{k\pi}$ for odd k:

$$\begin{aligned} f\left(t\right) &= \frac{6}{\pi}\sin\left(\frac{\pi t}{3}\right) + \frac{6}{3\pi}\sin\left(\frac{3\pi t}{3}\right) + \frac{6}{5\pi}\sin\left(\frac{5\pi t}{3}\right) + \frac{6}{7\pi}\sin\left(\frac{7\pi t}{3}\right) + \cdots \\ &= \frac{6}{\pi}\left[\sin\left(\frac{\pi t}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{3}\right) + \frac{1}{7}\sin\left(\frac{7\pi t}{3}\right) + \cdots\right] \end{aligned}$$

(ii) To deduce the given series we need to substitute $t = 3/2$. *What is the value of* $f\left(t\right)$ *when* $t = 3/2$*?*

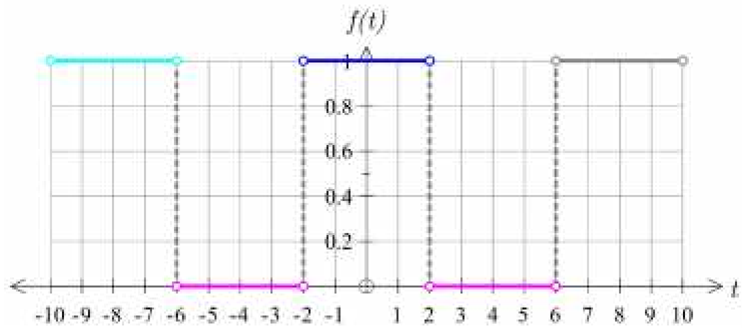
By the given graph we have $f\left(\dfrac{3}{2}\right) = \dfrac{3}{2}$. Substituting $t = 3/2$ into the above

Fourier series gives

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \frac{6}{\pi}\left[\sin\left(\frac{\pi}{\cancel{3}}\frac{\cancel{3}}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{\cancel{3}}\frac{\cancel{3}}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{\cancel{3}}\frac{\cancel{3}}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{\cancel{3}}\frac{\cancel{3}}{2}\right) + \cdots\right] \\ \frac{3}{2} &= \frac{6}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \cdots\right] \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \end{aligned}$$

This is our required result.

3. (i) The given waveform is:



This waveform has a period $2L = 8$ which gives $L = 4$. We have an even function so it only has the constant term and the cosine coefficients. *What is the constant term equal to?*

Remember the constant term A_0 represents the average value of the function over a complete period. *What is the average value of the above waveform between* -2 *and 6?*

1/2. Therefore $A_0 = 1/2$.

The cosine coefficients are given by

(17.22) $$A_k = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Substituting $L = 4$ into this formula gives

$$A_k = \frac{2}{4}\int_0^4 f(t)\cos\left(\frac{k\pi t}{4}\right)\mathrm{d}t \qquad (*)$$

What is $f(t)$ *equal to between 0 and 4?*

By the above graph we have

$$f(t) = \begin{cases} 1 & \text{when } 0 < t < 2 \\ 0 & \text{when } 2 < t < 4 \end{cases}$$

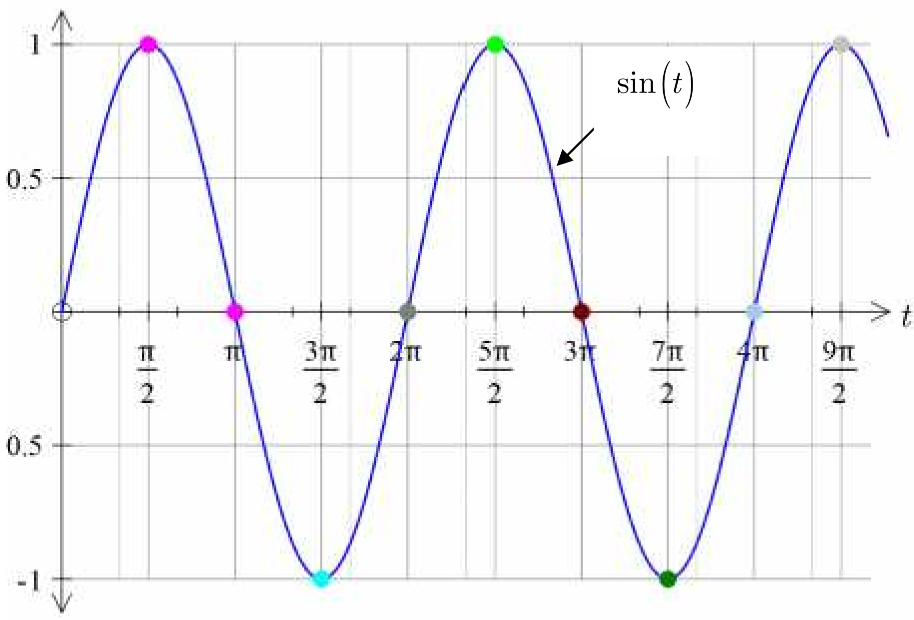
Putting this into (*) yields

$$\begin{aligned} A_k &= \frac{1}{2}\left[\int_0^2 (1)\cos\left(\frac{k\pi t}{4}\right)\mathrm{d}t + \int_2^4 (0)\cos\left(\frac{k\pi t}{4}\right)\mathrm{d}t\right] \\ &= \frac{1}{2}\int_0^2 \cos\left(\frac{k\pi t}{4}\right)\mathrm{d}t \\ &= \frac{1}{2}\left[\frac{\sin(k\pi t/4)}{k\pi/4}\right]_0^2 \qquad \left[\text{Using } \int \cos(mt)\,\mathrm{d}t = \frac{\sin(mt)}{m}\right] \\ &= \frac{4}{2k\pi}\left[\sin\left(\frac{2k\pi}{4}\right) - \sin(0)\right] \\ &= \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

We have

$$A_k = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) \qquad (\ddagger)$$

From the sine graph:



We have

$$\sin\left(k\frac{\pi}{2}\right)=\begin{cases}1 & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -1 & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

Putting this into (‡) yields

$$A_k=\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)=\begin{cases}2/k\pi & \text{if } k=1,\ 5,\ 9,\ldots\\ 0 & \text{if } k=\text{even}\\ -2/k\pi & \text{if } k=3,\ 7,\ 11,\ldots\end{cases}$$

The generic Fourier series of period $2L$ is given by

(17.20) $$f(t)=A_0+A_1\cos\left(\frac{\pi t}{L}\right)+A_2\cos\left(\frac{2\pi t}{L}\right)+\cdots+B_1\sin\left(\frac{\pi t}{L}\right)+B_2\sin\left(\frac{2\pi t}{L}\right)+\cdots$$

Remember we are given an even function so there are *no* sine terms (no B_k).

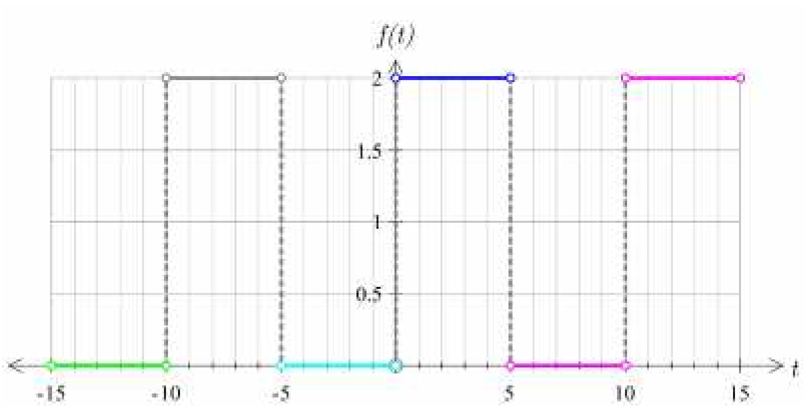
From above the constant term $A_0=1/2$.

Also note by the above formula for A_k that we only have odd cosine terms.

Substituting $L=4$ and evaluating the A_k terms gives

$$\begin{aligned}f(t)&=\frac{1}{2}+\frac{2}{\pi}\cos\left(\frac{\pi t}{4}\right)-\frac{2}{3\pi}\cos\left(\frac{3\pi t}{4}\right)+\frac{2}{5\pi}\cos\left(\frac{5\pi t}{4}\right)-\frac{2}{7\pi}\cos\left(\frac{7\pi t}{4}\right)+\cdots\\ &=\frac{1}{2}+\frac{2}{\pi}\left[\cos\left(\frac{\pi t}{4}\right)-\frac{\cos(3\pi t/4)}{3}+\frac{\cos(5\pi t/4)}{5}-\frac{\cos(7\pi t/4)}{7}+\cdots\right]\end{aligned}$$

4. We are asked to find the Fourier series of the following waveform:



This waveform has a period of $2L = 10$ which implies that $L = 5$.

It is not an odd or even function so we have to evaluate all the Fourier coefficients. *What is the average value of the above function between 0 and 10?*

1 so $A_0 = 1$. *How do we evaluate the cosine coefficients,* A_k *?*

(17.17) $$A_k = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Putting $L = 5$ and $f(t) = 2$ between 0 and 5 ($f(t) = 0$ between -5 and 0 so the integral is zero) gives

$$\begin{aligned} A_k &= \frac{1}{5}\left\{\int_0^5 \left[2\cos\left(\frac{k\pi t}{5}\right)\right]\mathrm{d}t + 0\right\} \\ &= \frac{2}{\cancel{5}}\left[\frac{\sin(k\pi t/5)}{k\pi/\cancel{5}}\right]_0^5 \\ &= \frac{2}{k\pi}\underbrace{\left[\sin(k\pi) - \sin(0)\right]}_{=0} = 0 \end{aligned}$$

Hence there are *no* cosine coefficients. Note that if there is an absence of cosine terms in the Fourier series then this does *not* imply that the function is odd as you can see from this example.

The sine coefficients are given by

(17.19) $$B_k = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Similarly we have

$$B_k = \frac{1}{5}\left\{\int_0^5 \left[2\sin\left(\frac{k\pi t}{5}\right)\right]\mathrm{d}t + 0\right\}$$
$$= \frac{2}{\not{5}}\left[-\frac{\cos\left(k\pi t/5\right)}{k\pi/\not{5}}\right]_0^5$$
$$= -\frac{2}{k\pi}\left[\cos\left(k\pi\right) - \cos\left(0\right)\right] = -\frac{2}{k\pi}\left[\cos\left(k\pi\right) - 1\right]$$

If k is even then $\cos\left(k\pi\right) = 1$ and the above B_k calculation is

$$B_k = -\frac{2}{k\pi}\underbrace{\left[1-1\right]}_{=0} = 0$$

If k is odd then $\cos\left(k\pi\right) = -1$ and the above B_k calculation is

$$B_k = -\frac{2}{k\pi}\left[-1-1\right] = \frac{4}{k\pi}$$

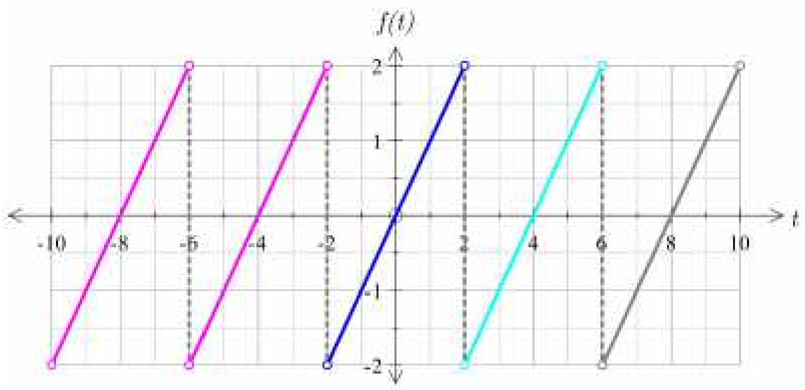
Substituting $A_0 = 1$, $A_k = 0$, $B_k = \frac{4}{k\pi}$ for odd k and $L = 5$ into the general Fourier series

(17.20) $$f\left(t\right) = A_0 + A_1\cos\left(\frac{\pi t}{L}\right) + A_2\cos\left(\frac{2\pi t}{L}\right) + \cdots + B_1\sin\left(\frac{\pi t}{L}\right) + B_2\sin\left(\frac{2\pi t}{L}\right) + \cdots$$

Gives

$$f\left(t\right) = 1 + \underbrace{0}_{\text{No Cosine terms}} + \frac{4}{\pi}\sin\left(\frac{\pi t}{5}\right) + 0 + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{5}\right) + 0 + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{5}\right) + \cdots$$
$$= 1 + \frac{4}{\pi}\left[\sin\left(\frac{\pi t}{5}\right) + \frac{\sin\left(3\pi t/5\right)}{3} + \frac{\sin\left(5\pi t/5\right)}{5} + \cdots\right]$$

5. We are given the following waveform:



This waveform is an odd function so the Fourier series only consists of the sine coefficients; B_k. *What is the period of the above waveform?*

Period $2L = 4$ so $L = 2$.

We use the following formula to find the sine coefficients:

$$B_k = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi t}{L}\right)\mathrm{d}t$$

Substituting $L = 2$ into this gives

$$B_k = \frac{\not{2}}{\not{2}}\int_0^2 f(t)\sin\left(\frac{k\pi t}{2}\right)\mathrm{d}t = \int_0^2 f(t)\sin\left(\frac{k\pi t}{2}\right)\mathrm{d}t \qquad (\dagger)$$

What is $f(t)$ equal to between 0 and 2?

$f(t) = t$. Substituting this into (†) gives

$$B_k = \int_0^2 \left[t\sin\left(\frac{k\pi t}{2}\right)\right]\mathrm{d}t \qquad (\dagger\dagger)$$

We need to use integration by parts to find this integral. Let

$$u = t \qquad v' = \sin\left(\frac{k\pi t}{2}\right)$$

$$u' = 1 \qquad v = \int \sin\left(\frac{k\pi t}{2}\right)\mathrm{d}t = -\frac{\cos(k\pi t/2)}{k\pi/2}$$

We have

$$\begin{aligned}\int_0^2 \left[t\sin\left(\frac{k\pi t}{2}\right)\right]\mathrm{d}t &= -\frac{2}{k\pi}\Big[t\cos(k\pi t/2)\Big]_0^2 + \frac{2}{k\pi}\int_0^2 \cos(k\pi t/2)\,\mathrm{d}t \qquad \left[\text{Using } \int uv'\mathrm{d}t = uv - \int u'v\ \mathrm{d}t\right]\\ &= -\frac{2}{k\pi}\Big[2\cos(\not{2}k\pi/\not{2}) - 0\Big] + \frac{2}{k\pi}\left[\frac{\sin(k\pi t/2)}{k\pi/2}\right]_0^2\\ &= -\frac{4}{k\pi}\cos(k\pi) + \frac{4}{k^2\pi^2}\underbrace{\Big[\sin(\not{2}k\pi/\not{2}) - \sin(0)\Big]}_{=0}\\ &= -\frac{4}{k\pi}\cos(k\pi)\end{aligned}$$

Substituting this into (††) yields

$$B_k = \int_0^2 \left[t\sin\left(\frac{k\pi t}{2}\right)\right]\mathrm{d}t = -\frac{4}{k\pi}\cos(k\pi)$$

Using the trigonometric result:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

We have

$$B_k = -\frac{4}{k\pi}\cos(k\pi) = \begin{cases} -4/k\pi & \text{if } k = \text{even} \\ 4/k\pi & \text{if } k = \text{odd} \end{cases}$$

Substituting $A_0 = 0$, $A_k = 0$, $B_k = \begin{cases} -4/k\pi & \text{if } k = \text{even} \\ 4/k\pi & \text{if } k = \text{odd} \end{cases}$ and $L = 2$ into the general Fourier series

$$\text{(17.20)} \quad f(t) = A_0 + A_1\cos\left(\frac{\pi t}{L}\right) + A_2\cos\left(\frac{2\pi t}{L}\right) + \cdots + B_1\sin\left(\frac{\pi t}{L}\right) + B_2\sin\left(\frac{2\pi t}{L}\right) + \cdots$$

Gives

$$\begin{aligned} f(t) &= 0 + 0 + \frac{4}{\pi}\sin\left(\frac{\pi t}{2}\right) - \frac{4}{2\pi}\sin(2\pi t/2) + \frac{4}{3\pi}\sin(3\pi t/2) - \frac{4}{4\pi}\sin(4\pi t/2) + \cdots \\ &= \frac{4}{\pi}\left[\sin\left(\frac{\pi t}{2}\right) - \frac{\sin(2\pi t/2)}{2} + \frac{\sin(3\pi t/2)}{3} - \frac{\sin(4\pi t/2)}{4} + \cdots\right] \end{aligned}$$