## Complete Solutions to Exercise 15a

1. (a) We have

$$\frac{2}{3}=0+\frac{2}{3}=0+\frac{1}{3/2}=0+\cfrac{1}{1+\cfrac{1}{2}}$$

Hence $\frac{2}{3}=[0;\ 1,\ 2]$.

(b) We need to convert $\frac{7}{5}$ into a simple continued fraction:

$$\frac{7}{5}=1+\frac{2}{5}=1+\frac{1}{5/2}=1+\cfrac{1}{2+\cfrac{1}{2}}$$

Therefore we have $\frac{7}{5}=[1;\ 2,\ 2]$.

(c) We are given $\frac{33}{7}$:

$$\frac{33}{7}=4+\frac{5}{7}=4+\frac{1}{7/5}=4+\cfrac{1}{1+\cfrac{2}{5}}=4+\cfrac{1}{1+\cfrac{1}{5/2}}=4+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}$$

We have $\frac{33}{7}=[4;\ 1,\ 2,\ 2]$.

(d) We need to convert $\frac{355}{113}$ into a simple continued fraction:

$$\frac{355}{113}=3+\frac{16}{113}=3+\frac{1}{113/16}=3+\cfrac{1}{7+\cfrac{1}{16}}$$

Therefore $\frac{355}{113}=[3;\ 7,\ 16]$.

(e) We are required to convert $\frac{181}{57}$ into a simple continued fraction:

$$\frac{181}{57}=3+\frac{10}{57}=3+\frac{1}{57/10}=3+\cfrac{1}{5+\cfrac{7}{10}}=3+\cfrac{1}{5+\cfrac{1}{10/7}}=3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{3}{7}}}$$

$$=3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{7/3}}}=3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}$$

We have $\frac{181}{57}=[3;\ 5,\ 1,\ 2,\ 3]$.

(f) We have to find the simple continued fraction of $-\frac{181}{57}$:

$$-\frac{181}{57}=-4+\frac{47}{57}=-4+\frac{1}{57/47}=-4+\cfrac{1}{1+\cfrac{10}{47}}$$

$$=-4+\cfrac{1}{1+\cfrac{1}{47/10}}=-4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{7}{10}}}=-4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{10/7}}}$$

$$=-4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{3}{7}}}}=-4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{7/3}}}}=-4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}}$$

We have $-\frac{181}{57}=[-4;\ 1,\ 4,\ 1,\ 2,\ 3]$.

2. (a) We are given the simple continued fraction $[1;\quad 2\quad 7\quad 3]$. This means

$$[1;\quad 2\quad 7\quad 3]=1+\cfrac{1}{2+\cfrac{1}{7+\cfrac{1}{3}}}$$

Evaluating the right hand side gives

$$1+\cfrac{1}{2+\cfrac{1}{7+\cfrac{1}{3}}}=1+\cfrac{1}{2+\cfrac{1}{22/3}}=1+\cfrac{1}{2+\cfrac{3}{22}}=1+\frac{1}{47/22}=1+\frac{22}{47}=\frac{69}{47}$$

Hence we have $[1;\quad 2\quad 7\quad 3]=\frac{69}{47}$.

(b) We need to convert $[1;\,2,\,5,\,1,\,1,\,2]$ into a rational number. *What does* $[1;\,2,\,5,\,1,\,1,\,2]$ *mean?*

$$[1;\,2,\,5,\,1,\,1,\,2]=1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

Evaluating this gives

$$1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}=1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{3/2}}}}=1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{2}{3}}}}=1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{5/3}}}=1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{3}{5}}}$$

Working out the last term on the right hand side:

$$1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{3}{5}}}=1+\cfrac{1}{2+\cfrac{1}{\frac{28}{5}}}=1+\cfrac{1}{2+\cfrac{5}{28}}=1+\cfrac{1}{\frac{61}{28}}=1+\frac{28}{61}=\frac{89}{61}$$

Therefore $[1;\ 2,\ 5,\ 1,\ 1,\ 2]=\dfrac{89}{61}$.

(c) We need to convert $[9;\ 2,\ 1,\ 1,\ 1,\ 2,\ 3]$ into a rational number:

$$[9;\ 2,\ 1,\ 1,\ 1,\ 2,\ 3]=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}}}$$

Working out this simple continued fraction gives

$$9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}}}=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{7/3}}}}}=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{3}{7}}}}}$$

$$=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{10}{7}}}}}=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{7}{10}}}}=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\frac{17}{10}}}}$$

$$=9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{10}{17}}}=9+\cfrac{1}{2+\cfrac{1}{\frac{27}{17}}}=9+\cfrac{1}{2+\cfrac{17}{27}}=9+\cfrac{1}{\frac{71}{27}}=9+\frac{27}{71}=\frac{666}{71}$$

We have $[9;\ 2,\ 1,\ 1,\ 1,\ 2,\ 3]=\dfrac{666}{71}$.

(d) We are required to find the rational number whose simple continued fraction is given by $[1;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2]$.

$$[1;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2]=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}}}$$

Evaluating the right hand side:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{3}{2}}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{2}{3}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{5}{3}}}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{3}{5}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{8}{5}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{5}{8}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{13}{8}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\frac{8}{13}}}=1+\cfrac{1}{1+\cfrac{1}{\frac{21}{13}}}=1+\cfrac{1}{1+\frac{13}{21}}=1+\cfrac{1}{\frac{34}{21}}=1+\frac{21}{34}=\frac{55}{34}$$

We have $[1;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2]=\dfrac{55}{34}$.

3. This time we need to use the Euclidean algorithm in each case.

(a) We are given $\dfrac{743}{450}$:

$$743=1(450)+293$$
$$450=1(293)+157$$
$$293=1(157)+136$$
$$157=1(136)+21$$
$$136=6(21)+10$$
$$21=2(10)+1$$
$$10=10(1)+0$$

We can write the given fraction $\dfrac{743}{450}$ as a continued simple fraction by writing the quotients in order of the above derivation:

$$\frac{743}{450}=[1;\ 1,\ 1,\ 1,\ 6,\ 2,\ 10]$$

(b) This time we are given $\dfrac{743}{459}$:

$$743 = 1(459) + 284$$
$$459 = 1(284) + 175$$
$$284 = 1(175) + 109$$
$$175 = 1(109) + 66$$
$$109 = 1(66) + 43$$
$$66 = 1(43) + 23$$
$$43 = 1(23) + 20$$
$$23 = 1(20) + 3$$
$$20 = 6(3) + 2$$
$$3 = 1(2) + 1$$
$$2 = 2(1)$$

Reading off the quotients gives

$$\frac{743}{459} = [1;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 6,\ 1,\ 2]$$

(c) We are given $\frac{89}{301}$. Using the Euclidean Algorithm:

$$89 = 0(301) + 89$$
$$301 = 3(89) + 34$$
$$89 = 2(34) + 21$$
$$34 = 1(21) + 13$$
$$21 = 1(13) + 8$$
$$13 = 1(8) + 5$$
$$8 = 1(5) + 3$$
$$5 = 1(3) + 2$$
$$3 = 1(2) + 1$$
$$2 = 2(1) + 0$$

We have $\frac{89}{301} = [0;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2]$.

(d) Using the Euclidean algorithm for $\frac{103993}{33102}$:

$$103993 = 3(33102) + 4687$$
$$33102 = 7(4687) + 293$$
$$4687 = 15(293) + 292$$
$$293 = 1(292) + 1$$
$$292 = 292(1)$$

Therefore $\frac{103993}{33102} = [3;\ 7,\ 15,\ 1,\ 292]$.